High-Frequency Tail Risk Premium and Stock Return Predictability*

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March 20, 2022

Abstract

We propose a novel measure of the market return tail risk premium based on minimum-distance state price densities recovered from high-frequency data. The tail risk premium estimate extracted from S&P 500 returns predicts the market equity and variance risk premiums and expected excess returns on a cross section of characteristics-sorted portfolios. Additionally, we describe the differential role of the quantity of tail risk, and of the tail premium, in shaping the future distribution of index returns. The results are robust to including established measures of tail and variance risk, and of risk premiums, in the predictive models.

Keywords: Tail Risk, Risk-Neutral Measure, Expected Shortfall, Intra-day Market Returns.

JEL Code: G12, G13, G17.

*We would like to Thank Gustavo Freire and Rodrigo Canales for useful suggestions. We also thank seminar participants at Kellogg School of Management, the 2017 SoFiE Conference in New York, the 2017 Vienna-Copenhagen Conference on Financial Econometrics, and the 2018 IAAE Meeting in Montreal for their comments and suggestions. The second author acknowledges financial support from ANBIMA and FAPERJ. The third author thanks the NSERC, the SSHRC and the FQRSC research grant agencies for their financial support. He is a TSE associate faculty and a research Fellow of CIRANO and CIREQ.

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1 Introduction

Compensation for tail risk plays a prominent role in explaining the equity and the variance risk premiums. This stylized fact is supported by numerous studies of index option markets (Bollerslev et al. (2015); Andersen et al. (2017)), which conclude that the market tail risk premium predicts index returns at different horizons, while such returns are not explained by market volatility. Option market data is typically used to quantify the tail risk under the equivalent martingale (pricing) measure.

In this paper, in contrast, we put forward an alternative option-free method of quantifying the tail risk premium and show its empirical strength. We compute the tail risk premium at a daily frequency using intraday return data on the S&P 500 index. In an extensive empirical study, we show that our tail premium outperforms other related measures in terms of predictive power for equity and variance risk premiums of index returns, and for excess returns of characteristic-sorted equity portfolios.

We estimate the tail risk premium, denoted as $\Delta_{\mathbb{Q}}^{\mathbb{P}} ES$, as the difference between the expected shortfall calculated under the risk-neutral ($ES^{\mathbb{Q}}$) and physical ($ES^{\mathbb{P}}$) probability measures estimated from intraday data. Therefore, we estimate conditional daily tail risk premiums based on the most recent high-frequency information. The intraday risk-neutralization is based on a nonparametric adjustment of the raw index returns. Motivated by Ait-Sahalia and Lo (1998, 2000), the risk adjustment puts a higher probability weight on extreme negative returns, reflecting investors' compensation for bad states of the world.

Bollerslev et al. (2015) document that time variation in the quantity of left-tail risk under the pricing measure has significant predictive ability for the distribution of future stock returns and explains a large portion of the variance risk premium. They summarize the quantity of tail risk in an estimate of the shape parameter of the tail of the return distribution.[1] Since $ES^{\mathbb{P}}$ is a conditional average of extreme negative returns, it maps directly

[1]The tail of a probability distribution can be completely characterized via Central Limit Theorems by a so-called shape parameter that determines the tail thickness. Building on these ideas, Bollerslev and Todorov (2014) advocate for asymmetric and time-varying shape parameters when modeling financial

to the value of this parameter (see Section 2.4). Then, since our estimate of the tail risk premium captures the time-variation of the risk-neutral and physical tail shape parameters of S&P500 returns, it should forecast stock returns and risk premiums.

From an economic viewpoint, $ES^{\mathbb{P}}$ is the expectation of the payoff of an out-of-the-money (OTM) put option with extremely short maturity, naturally sensitive to negative jumps in returns. Thus, our measure can also be interpreted as the expected gain from selling a short-maturity OTM put option. The tail premium becomes higher when negative jumps in returns trigger investors' perception of an increase in market downside risk. This negative perception implies larger risk-neutral probabilities on the tail and consequently a larger gap between the risk neutral and physical put prices. Therefore, if aversion to downside risk is priced, we expect our measure to drive risk premiums.[2]

We examine the predictive power of $\Delta^{\mathbb{P}}_{\mathbb{Q}}ES$ in empirical tests. Due to the high-frequency nature of the measure, we focus our predictive exercises on very short horizons, directly complementing the longer-horizon predictability results documented in the literature. In the predictive regressions, we include other established tail and variance risk measures as controls. Among these, three are particularly important: the physical expected shortfall ($ES^{\mathbb{P}}$), the Left Tail Variance of Bollerslev et al. (2015) (LTV), and the Variance Risk Premium (VRP). $ES^{\mathbb{P}}$, based on observed intraday returns of the S&P500 index, is a measure of realized tail risk. Thus, as our tail risk premium is the difference of ES under $\mathbb{Q}$ and $\mathbb{P}$, including $ES^{\mathbb{P}}$ allows us to assess what better predicts stock risk premiums, the quantity of tail risk, or the tail risk premium. LTV, since directly built from short-maturity options, provides a good benchmark to assess if options contain information beyond that provided by our measure. The VRP has been shown to be an important predictor of equity risk premium (see Bollerslev et al. (2009)). We complement these three measures with the realized variance of the index intraday returns (RV) calculated from return data sampled at

markets, identifying significant temporal variation of both (left and right) tails of S&P 500 returns.

[2]Andersen et al. (2015) indeed demonstrate that once triggered, the perception of left-tail risk among investors remains elevated for long periods, even though the likelihood of such events quickly decays under the statistical probability measure, which in turn results in a significant market tail risk premium.

the five-minute frequency, and with other measures of realized return variation.

In the predictive analysis of the equity premium, our measure $\Delta_{\mathbb{Q}}^{\mathbb{P}}ES$ is the only significant predictor at the 1-day horizon, at a 1% significance level, with a solo R^2 of 0.7%. The multivariate regression that includes $\Delta_{\mathbb{Q}}^{\mathbb{P}}ES$, LTV, the realized variance RV and the variance risk premium VRP has a non-negligible R^2 of 2%. In contrast, for the weekly and monthly horizons, no predictor is significant. Our discovery of short-horizon equity premium predictability is an important contribution to the literature as such predictability is generally harder to detect since the signal-to-noise ratio is very low. $\Delta_{\mathbb{Q}}^{\mathbb{P}}ES$ also succesfully predicts the variance risk premium at multiple horizons: together with the lagged VRP, it delivers an R^2 of approximately 30%.

We also analyze the relationship between our selected predictive candidate variables (tail risk and variance measures) and one-day-ahead excess returns on portfolios comprised of stocks sorted on various characteristics: market capitalization, book-to-market value, profitability, investment, momentum, reversal and industry. A salient feature of this predictive regression analysis is that $\Delta_{\mathbb{Q}}^{\mathbb{P}}ES$ is the only variable that predicts the returns on most of the portfolios with high t-stats and economically significant R^2s (between 0.5 and 1.4 %). Adding the four other measures ($ES^{\mathbb{P}}$, LTV, RV and VRP) one at a time does not meaningfully improve the R^2. We conclude that the tail risk premium is the central determinant of the time variation of the premiums in equities at short horizons.

Our analysis of the predictive power of $\Delta_{\mathbb{Q}}^{\mathbb{P}}ES$ and $ES^{\mathbb{P}}$ for risk premiums suggests that it is the tail premium rather than the level of tail risk that contains relevant predictive information. This result is largely in line with Bollerslev and Todorov (2011), Bollerslev et al. (2015) and Andersen et al. (2017), among others, who study the risk-return trade-off concluding that it is the variance risk premium rather than the quantity of risk (expected variance) that is informative of future risk premiums. Thus, in order to better understand the power of our tail measure in predicting risk premiums measures, we analyze its impact on the future distribution of S&P500 returns.[3] We assume that the predictors ($\Delta_{\mathbb{Q}}^{\mathbb{P}}ES$, $ES^{\mathbb{P}}$, LTV, RV, VRP) are the main factors driving the variation of the next-day return distribution. We estimate the quantile regression model of Koenker and Gilbert (1978) using data from 2004 to 2018 and identify that while $\Delta_{\mathbb{Q}}^{\mathbb{P}}ES$ has a fundamental role in forecasting the center and right quantiles of the S&P500 return distribution, $ES^{\mathbb{P}}$ has strong forecasting ability for the left and right tails and VRP for right extreme quantiles. This informs that risk and risk premium measures have a complementary role in determining the distribution of future returns. While $ES^{\mathbb{P}}$, a measure of risk, can better determine the extreme movements of returns, the risk-premium measures ($\Delta_{\mathbb{Q}}^{\mathbb{P}}ES$ and VRP) have a stronger role on how positive (above median) returns affect the equity premium. We then conduct a thorough out-of-sample evaluation of interval forecasts using all of the tests prescribed by Christoffersen (1998), and using the last 25% of our sample as the out-of-sample period. Our quantile model fares very well with marginal rejection in the center of the distribution, while not being rejected for the whole distribution.

The rest of the paper is organized as follows. Section 1.1 offers a brief review of the related literature. Section 2 describes how we construct and estimate our tail risk premium measure based on market returns. Section 3 describes our empirical predictability results for returns and variance risk premiums and for the quantiles of the S&P 500 return distribution. Out-of-sample tests of interval forecast conditional coverage, error independence and unconditional coverage are also discussed. Section 4 presents various robustness tests varying different parameters used to estimate our measure. We summarize and put in perspective our contributions in section 5.

1.1 Related Literature

Our paper is related to the rich and growing literature on the estimation of tail risk and systematic risk measures and their use to predict the equity and variance risk premiums. This

[3] This analysis can help to shed light on the mechanism of the increase in the premium that is associated with the variation in $\Delta_{\mathbb{Q}}^{\mathbb{P}}ES$: can it predict more accurately low, intermediate, or large returns?

includes, among others, Bali et al. (2009), Allen et al. (2012), Siriwardane (2013), Kelly and Jiang (2014), Adrian and Brunnermeier (2016), Brownlees and Engle (2017). In particular, Bollerslev et al. (2015) decompose the variance risk premium and examine the importance of the diffusive and jump components for return predictability. Andersen et al. (2017) estimate the variation in the left tail of the return distribution from short-maturity options with significant predictive power for future short-term returns. Andersen et al. (2020) extend this evidence to international equity markets. Vilkov and Xiao (2013), Ghysels and Wang (2014), and Huggenberger et al. (2018) use daily index options to model forward looking tail risk based on Value-at-Risk and/or Expected Shortfall measures. In contrast to these papers, we are the first to provide a methodology to compute the tail risk premium in high-frequency environments that is applicable to virtually any set of returns.

To estimate the risk-neutral leg of our tail risk premium measure, we build on Almeida et al. (2017), who compute a nonparametric risk-neutral expected shortfall based on a cross-section of daily portfolio or security returns.[4] In contrast to that paper, we rely solely on returns on a broad market index and we use high-frequency intra-day data to obtain more information about the tail. Moreover, while in the aforementioned paper the authors propose the risk-neutral expected shortfall as a new tail risk measure, we concentrate on measuring the tail risk premium.[5]

Our paper is close in spirit to Weller (2019) who develops a real-time tail risk measure based on intra-day bid and ask quotes. While Weller (2019) focuses on the natural relationship between tail risk and jumps, providing substantial evidence of jump realization predictability using intra-day data, we focus on the broader relationship between tail risk, market returns and the cross sections of characteristic-based portfolios. Additionally, Weller

[4]Kelly and Jiang (2014) also use a large cross-section of observed returns to compute a tail risk measure by assuming that asset return tails follow a power law. With particular emphasis on the financial sector, Allen et al. (2012) and Brownlees and Engle (2017) adopted VaR and Expected Shortfall measures to estimate systemic risks. In this literature, the estimated tail risk measures are calculated on a monthly or weekly basis rather than daily.

[5]In addition, by focusing on high-frequency individual returns instead of cross-sectional returns, supplementary economic restrictions like the non-negativity of the equity premium can be naturally imposed on the Euler equations as advocated by Campbell and Thompson (2008) and Pettenuzzo et al. (2014).

(2019)'s measure considers a panel of 2800 firms for its baseline one-factor model, while our measure necessitates only one day of intra-daily observations on one stock index. It can be easily applied to different markets and assets for which intra-day returns, at any frequency, are available, avoiding the need to rely on option prices or bid-ask spreads.

Our paper also complements the findings obtained with the use of high-frequency factor models in Bollerslev et al. (2013), Bollerslev et al. (2016) and Aït-Sahalia et al. (2020). Bollerslev et al. (2013) use a large high-frequency data set on the cross section of stock returns to measure the quantity of tail risk (jump tails) under the physical probability measure. In contrast, we identify a measure of the high frequency tail risk premium that captures the wedge between the risk-neutral and physical worlds. Bollerslev et al. (2016) consider an extension of the CAPM model in which a beta on the jump component of the market return is studied together with the beta on the continuous component of the market return. They found that the only significant premium is associated with the jump component. We offer a new tail risk premium measure that is not dependent on options data and does not rely on any parametric dynamics for the market return. Aït-Sahalia et al. (2020) estimate a multi-factor model at high-frequency and find that a large part of the market equity premium is due to exposures to the market's jump risk component and that various jump risks in Fama-French and momentum factors supersede their continuous counterparts as the primary pricing factors. The predictive power of our tail risk premium $\Delta_{\mathbb{Q}}^{\mathbb{P}}ES$ for the expected returns of the characteristic-based portfolios is consistent with their findings and with the story that our measure captures time variation of the shape parameter of the tail of returns.

2 The Nonparametric Tail Risk Premium

2.1 Background

Let $(\Omega, \mathbb{F}, \mathbb{P})$ be a probability space (with $\mathbb{P}$ the physical probability measure), where R and R_F are random variables denoting, respectively, the return of a primitive basis asset (the stock index), and a short risk-free rate. An admissible risk-neutral distribution (RND) $\mathbb{Q}$ is represented by a density q, a non-negative random variable with unitary mean satisfying the Euler pricing equation for the index returns:[6]

$$\mathbb{E}^{\mathbb{Q}}(R - R_F) \equiv \mathbb{E}^{\mathbb{P}}(q(R - R_F)) = 0, \tag{1}$$

$$\text{with } \mathbb{E}^{\mathbb{P}}(q) = 1,\ q \geq 0 \tag{2}$$

At each date t, we observe a sample $\{R_i^t\}_{i=1,...,T}$ of high-frequency stock index returns ($T > 1$). We use this high-frequency time series to identify a conditional RND $\mathbb{Q}_t$ via its density q^t and the empirical conditional physical distribution $\mathbb{P}_t$, with density $p_i^t = \frac{1}{T}, i = 1, ..., T$, for all t.[7]

2.2 Definition

Our objective is to build a simple tail risk premium measure that depends solely on the returns of the single stock index observed at a high frequency. To that end, for each date t, we estimate daily expected shortfalls under the conditional physical and risk-neutral measures:

[6] A RND is a probability distribution $\mathbb{Q}$ equivalent to the physical distribution $\mathbb{P}$, under which the basis assets are correctly priced, i.e, satisfy the Euler equation. It can be represented one-to-one with its corresponding density q. In the paper, we use these two definitions interchangeably.

[7] We assume stationarity and ergodicity of the composite process $(q_i^t, R_i^t)_{\{i=1,...T\}}$, such that it satisfies a time-series version of the law of large numbers (Hansen and Richard, 1987).

$$ES_t^{\mathbb{P}_t} := \mathbb{E}^{\mathbb{P}_t}[(s_\alpha - R)^+] \tag{3}$$

$$ES_t^{\mathbb{Q}_t} := \mathbb{E}^{\mathbb{Q}_t}[(s_\alpha - R)^+] \tag{4}$$

In the equation above, t is the estimation date, α is a confidence level, s_α is the α-quantile of R under the physical probability, and $\mathbb{P}_t$, and $\mathbb{Q}_t$ indicate the physical and risk-neutral conditional probability densities at time t. Note that we use a version of ES that can also be interpreted as a put option payoff with strike s_α. The random variable $(s_\alpha - R)^+$ is positive only in the states of nature where $R < s_\alpha$, which for $\alpha < 0.5$, at least for symmetric distributions, occurs for returns R smaller than the risk-free rate R_F, i.e., negative excess returns. The smallest the α, the further we are in the left tail of R. We adopt ES to measure risk since it is a coherent measure of risk (Artzner et al., 1999), which overcomes the main deficiencies of the Value-at-Risk (VaR) measure. In particular, while VaR completely ignores the behavior of returns in the tail beyond its confidence level, ES takes an average of these tail returns being highly sensitive to what happens in the tail.[8]

Our tail risk premium measure is defined as the difference between these two expected shortfalls:

$$\Delta_{\mathbb{Q}}^{\mathbb{P}} ES_t = ES_t^{\mathbb{Q}_t} - ES_t^{\mathbb{P}_t} \tag{5}$$

Note that we keep the same threshold s_α for the two ES's to imply a tail risk premium that depends only on how investors' perception of market downside risk encoded in risk neutral probabilities makes $\mathbb{E}^{\mathbb{Q}_t}$ exceed its physical counterpart $\mathbb{E}^{\mathbb{P}_t}$. In our empirical application, we compute $\Delta_{\mathbb{Q}}^{\mathbb{P}} ES_t$ at a daily frequency using the intraday stock index returns. Since there is no overlapping of data when calculating our measure, it avoids spurious persistence and

[8]See Berkowitz and O'Brien (2002) and Jorion (2019) for examples in which VaR leads to overestimation of risks in calm periods, and underestimation during crisis.

is able to quickly react to the arrival of new information embedded in market returns.

At this point, our main challenge lies in finding a way of identifying the conditional risk-neutral probability distribution $\mathbb{Q}_t$ without using options in the estimation process. Following Almeida et al. (2017) who compute risk-neutral probability distributions from a cross section of portfolio returns, we compute $\mathbb{Q}_t$ at each date t by solving a specific minimum distance problem between the conditional physical probability distribution $\mathbb{P}_t$ and the family of risk neutral distributions that correctly price the S&P 500 excess returns $\{R_i^t\}_{i=1,...T}$ within day t. The details of this procedure are developed in the next subsection.

2.3 Identifying Conditional Risk-Neutral Densities from S&P 500 Returns

We work with a sequence of repeated one-period models indexed by time t, characterized by the high-frequency returns $\{R_i^t\}_{i=1,\ldots,T}$ sampled at t. Since $T > 1$, the market within each of these models is inherently incomplete and, under the assumption of no-arbitrage, there exists an infinity of RNDs pricing the index returns at each date t. Almeida and Garcia (2017) suggest identifying a subset of RNDs by minimizing functions in the Cressie Read family of discrepancies $\phi^\gamma(\pi) = \frac{\pi^{\gamma+1}-1}{\gamma(\gamma+1)}, \gamma \in \mathbb{R}$, that measure the distance between admissible RNDs $\mathbb{Q}$ with density q and the physical probability distribution $\mathbb{P}$ with density p. Each discrepancy $\phi^\gamma(q)$ allows for the identification of a specific RND q_*^γ with unique sensitivity (exposure) to higher order moments of the stock index returns.[9]

The minimum-discrepancy (MD) problem proposed by Almeida and Garcia (2017) for SDFs, re-adapted by Almeida and Freire (2022) for RNDs, in its sample version for a single basis asset can be stated as:

[9]The original analysis in Almeida and Garcia (2017) is performed in terms of Stochastic Discount Factors. Almeida and Freire (2022) adapt it in details to consider RNDs instead. Most of the technical details following below were originally derived in these two papers.

$$
\begin{aligned}
\hat{q} = \underset{\{q_1,\ldots,q_T\}}{\arg\min} \quad & \sum_{i=1}^{T} p_i \phi^{\gamma}\left(\frac{q_i}{p_i}\right) \\
\text{subject to} \quad & \sum_{i=1}^{T} q_i (R_i^t - R_F) = 0 \\
& \sum_{i=1}^{T} q_i = 1 \\
& q_i \geq 0 \, (\text{or } q_i > 0) \; \forall \, i
\end{aligned} \tag{6}
$$

where the last inequality depends on the discrepancy $\phi^{\gamma}(.)$ chosen to measure the distance between the RND q and the physical density p: If $\gamma > 0$, then $q \geq 0$ otherwise $q > 0$. In this paper, we assume homogeneous empirical probabilities $p_i = \frac{1}{T}, i = 1, 2, ..., T$ to represent the physical distribution, for all dates t.[10]

The MD problem (6) is computationally simpler and faster to solve in its dual formulation:

$$
\hat{\lambda} = \underset{\alpha \in \mathbb{R},\, \lambda \in \Lambda}{\arg\sup} \; \alpha + \sum_{i=1}^{T} p_i \phi^{*,\gamma}(\alpha + \lambda(R_i^t - R_F)) \tag{7}
$$

where $\Lambda \subseteq \mathbb{R}$ and $\phi^{*,\gamma}$ denotes the convex conjugate of ϕ^{γ}, restricted to a subset of the non-negative real line:

$$
\phi^{*,\gamma} = \underset{w \in [0,\infty) \cap \text{domain } \phi^{\gamma}}{sup} \; zw - \phi(w) \tag{8}
$$

In this dual problem, α and λ are Lagrange multipliers arising from the restrictions defining an admissible RND in (6). The multiplier determining that q has unitary mean (i.e., is a discrete probability distribution), α, can be concentrated out of the problem, while λ, the multiplier enforcing that returns satisfy the Euler equation, is the main one completely characterizing the RND. Below, we associate this dual problem to a marginal investor (or pseudo-investor) in the S&P 500 market solving an optimal portfolio problem selecting the

[10]The homogeneous probability assumption for $\mathbb{P}$ does not affect the key insights we are able to derive from this methodology. Alternatively, we could obtain the physical probabilities, for instance, using a kernel density estimator without introducing any additional complexity to the method.

weights on the risk-free rate and risky asset.

Cressie Read discrepancies indexed by negative γs are more sensitive to skewness and kurtosis of basis assets returns. Since our goal is to identify an RND to calculate the tail risk premium, working with a Cressie Read member indexed by a negative γ is an adequate choice. Almeida et al. (2017) work with the Hellinger discrepancy ($\gamma = -\frac{1}{2}$) to identify the RND from a cross section of stock returns, due to the robustness of this estimator reported in former econometric studies. In this paper, we choose instead to conduct our baseline analysis using the $\gamma = -3$ estimator since it is more sensitive to the skewness and the kurtosis of returns. We adopt the $\gamma = -\frac{1}{2}$ analysis later in Section 4, which discusses the robustness of our results. The predictive results obtained with the two estimators are similar with some minor differences discussed in that section.

For $\gamma < 0$ and equipped with a data sample, we have to solve the following dual optimization problem to obtain λ and q:

$$\hat{\lambda} = \arg\sup_{\lambda \in \Lambda} \; -\sum_{i=1}^{T} p_i \frac{1}{(\gamma+1)} \left(1 + \gamma\lambda\left(R_i^t - R_F\right)\right)^{\frac{\gamma+1}{\gamma}}, \tag{9}$$

$$\hat{q}_i^{\gamma} = \frac{\left(1 + \gamma\hat{\lambda}\left(R_i^t - R_F\right)\right)^{\frac{1}{\gamma}}}{\sum_{i=1}^{T} p_i \left(+\gamma\hat{\lambda}\left(R_i^t - R_F\right)\right)^{\frac{1}{\gamma}}}. \tag{10}$$

The above dual problem can be interpreted as a HARA utility optimal portfolio problem of an agent choosing an investment on a risky asset and on the risk-free rate (see Almeida and Freire (2022) for details). Provided that the equity risk premium $\mathbb{E}(R - R_F) > 0$, the optimal solution will contemplate buying a certain amount of the stock index. In such case, for any $\gamma < 0$, the RNDs q^{γ} put higher probability mass to any state of negative index return than the original physical distribution p. This guarantees that the risk-neutral leg of our tail risk premium measure, $ES^{\mathbb{Q}}$, is always greater or equal to the corresponding physical leg, $ES^{\mathbb{P}}$, implying that $\Delta_{\mathbb{Q}}^{\mathbb{P}} ES_t \geq 0$, $\forall t$. That is, if we restrict the equity premium to be non-negative, our tail risk premium will be a non-negative measure that peaks exactly on the days where the probability mass of the risk neutral measure is higher on the states of negative S&P500 returns. In our baseline analysis we restrict $\mathbb{E}(R - R_F) \geq 0$ and discuss in the robustness section the unrestricted case.

To understand the effect of the estimated RND on our tail risk premium measure, we replicate the Taylor expansion of the expected value of the discrepancy $\phi(\pi) = \frac{\pi^{\gamma+1}-1}{\gamma(\gamma+1)}$, around the RND mean 1 executed in Almeida and Freire (2022):

$$\mathbb{E}(\phi^{\gamma}(\pi)) = \frac{1}{2}E(\pi-1)^2 + \frac{\gamma-1}{3!}E(\pi-1)^3 + \frac{(\gamma-1)(\gamma-2)}{4!}E(\pi-1)^4 + ... \quad (11)$$

There are important aspects to observe about the weights attributed to skewness and kurtosis of the RND by each discrepancy function from Cressie Read family. First, all discrepancies indexed by negative gammas imply estimated RNDs that are sensitive to higher-order moments in the space of RNDs, which directly translates into sensitivity to higher moments of the stock index returns. The absolute weight given to kurtosis is smaller than that given to skewness for all $-2 < \gamma < 0$, and for $\gamma < -2$ the opposite happens. In the whole $\gamma \leq 0$ region, the weight given by the discrepancy to the skewness of a RND is negative, while the weight assigned to kurtosis is positive. Since we minimize the expected value of the discrepancy $E(\phi^{\gamma}(\pi))$ to identify our RNDs, they will be positively skewed and have minimal kurtosis being compatible with a "preference" for positive skewness and "aversion" to kurtosis of stock index returns, characteristics which are in line with the findings of Kraus and Litzenberger (1976) and Backus et al. (2011). For a detailed analysis of the relationship between Cressie Read discrepancies, RNDs and higher moments of returns see Almeida and Freire (2022).

2.4 Extreme Value Theory, Expected Shortfall and the Tail Shape Parameter

Extreme Value Theory (EVT) shows that once we are looking far in the tail of a random variable X with a given distribution F, at values larger than an exogenous threshold u, the conditional distribution function $F(X \leq x \mid x \geq u)$ can be well approximated by a Generalized Pareto Distribution (GPD) $G(\xi, \beta)$, defined below.[11] We use this result to identify a direct link between our ES measures of risk defined under $\mathbb{P}_t$ and $\mathbb{Q}_t$ and the corresponding shape parameters $\xi_t^{\mathbb{P}}$ and $\xi_t^{\mathbb{Q}}$ that determine the thickness of the GPD's that capture the probabilistic behavior of the tails of $\mathbb{P}_t$ and $\mathbb{Q}_t$.

The cumulative GPD distribution function is given by $G_{\xi,\beta}(x) = 1 - (1 + \frac{\xi x}{\beta})^{-\frac{1}{\xi}}, x \geq 0, \beta > 0, 0 < \xi < 1$, where β is a scaling parameter and ξ the shape parameter. The larger the ξ the thicker the tail is. It is well-known that given a sample of returns $\mathcal{R} = \{R_1, ...R_T\}$, whose conditional tail distribution is approximated by a GPD, and a fixed confidence level α, there is a direct relationship between the Expected Shortfall measure with confidence level α, $ES^{\alpha}(\mathcal{R})$, the Value at Risk based on the same confidence level, $VaR^{\alpha}(\mathcal{R})$, and the shape parameter ξ:

$$ES^{\alpha}(\mathcal{R}) = \frac{VaR^{\alpha}(\mathcal{R})}{1-\xi} + \frac{\beta - \xi u}{1-\xi} \tag{12}$$

Our tail risk measures ($ES_t^{\mathbb{P}_t}$, $ES_t^{\mathbb{Q}_t}$) and tail risk premium ($\Delta_{\mathbb{Q}}^{\mathbb{P}} ES_t$) are calculated on a daily frequency. Following Bollerslev and Todorov (2014), we allow the tail's shape parameters ($\xi_t^{\mathbb{P}}$, $\xi_t^{\mathbb{Q}}$) to be time-varying. Assuming for simplicity that the threshold u and scale β parameters are time-invariant and common to both physical $\mathbb{P}$ and risk-neutral distributions $\mathbb{Q}$, we can invert (12) to obtain the time-varying shape parameters as hyperbolic functions of our ES measures:

$$\xi_t^{\mathbb{P}} = 1 + \frac{VaR_t^{\mathbb{P},\alpha} + \beta - u}{u - ES_t^{\mathbb{P},\alpha}} \tag{13}$$

[11] This threshold-exceedance distribution, or distribution of the tail, is usually identified based on a sample of observed random variables with the same distribution F. See Chapter 7 in McNeil et al. (2005) for more details on EVT.

$$\xi_t^{\mathbb{Q}} = 1 + \frac{VaR_t^{\mathbb{Q},\alpha} + \beta - u}{u - ES_t^{\mathbb{Q},\alpha}} \tag{14}$$

Equation (12) also implies that our tail risk premium is a continuous function $H(.,.)$ of the tail shape parameters of both physical and risk neutral distributions:

$$\Delta_{\mathbb{Q}}^{\mathbb{P}} ES_t = H(\xi_t^{\mathbb{P}}, \xi_t^{\mathbb{Q}}) \tag{15}$$

Bollerslev and Todorov (2014) and Bollerslev et al. (2015) propose the left jump tail variance (LTV), a risk-neutral jump tail risk measure for the S&P 500 returns that is intuitive and has strong ability to forecast future returns of a cross section of stocks. They build a new dynamic model for the S&P 500 prices under the risk neutral measure $\mathbb{Q}$, whose dynamics is decomposed into a continuous stochastic volatility component and a jump component with time-varying stochastic jump-intensity.[12] The key novelty with respect to the previous literature is the use of EVT to model the tails (both left and right) of the jump-intensity measure, suggesting that they follow Frechet distributions with time-varying shape/decay parameters $F(x) = x^{-\alpha_t}, \alpha_t > 0$. They adopt the inverse of this shape parameter, $\xi_t = \frac{1}{\alpha_t}$, as a measure of tail risk. Since the Frechet distribution is directly comparable to the GPD when the shape parameter ξ is positive (i.e., $\xi_t > 0$), equations (14) and (15) directly connect our risk-neutral ES ($ES^{\mathbb{Q}}$) and tail risk premium ($\Delta_{\mathbb{Q}}^{\mathbb{P}} ES$) measures to LTV.[13]

3 Empirical results

Our empirical study of the predictability of risk premiums associated with the U.S. market index, identified by the S&P 500 index, is composed of three parts. The first part focuses

[12] LTV is estimated with the use of options. It represents the expected (risk-neutral) return volatility that stems from large negative price jumps. To identify and separate the diffusive part and the jump component of the return process, the authors use short-dated derivative securities on the index.

[13] Moreover, since $ES_t^{\mathbb{Q}}$ is an invertible function of $\xi_t^{\mathbb{Q}}$ and $\Delta_{\mathbb{Q}}^{\mathbb{P}} ES_t$ is linear in $ES_t^{\mathbb{Q}}$, LTV and our tail measures, at least from a theoretical point of view, induce the same probability filtration (i.e., the same information sets) along the time dimension. In empirical work they might differ since estimation is based on different sets of assets.

on the predictive power of our tail risk premium ($\Delta_{\mathbb{Q}}^{\mathbb{P}} ES$) and tail risk realization ($ES^{\mathbb{P}}$) measures for the market risk premium and the market variance risk premium. In part two, we extend the study to the cross-sectional predictive ability of our measures for the tail risk premium and tail risk realizations. The third part examines the broader implications for the predictability of the density of market index returns.

3.1 Dataset Description

Our dataset is compiled from a number of data sources and covers the period from January 2004 to December 2018. First, we obtain high-frequency data on the S&P 500 index from www.tickdata.com, and down-sample it to the five-minute frequency by registering the last observation in each five-minute window. This data is used not only to estimate $\Delta_{\mathbb{Q}}^{\mathbb{P}} ES$ and $ES^{\mathbb{P}}$, but also a number of high-frequency return variation measures, such as realized variance (RV), integrated quadratic variation (IV) and jump quadratic variation (JV).[14] Data on the VIX index is obtained from the CBOE via WRDS, whereas the data on daily close-to-close S&P 500 returns, inclusive of dividends, is obtained from CRSP. The risk-free rates and data on the cross-sectional stock portfolios is obtained from the Kenneth French Data Library.[15] High-frequency data on the options on the S&P 500 index is sourced from the CBOE in the form of best available bid and ask price quotes at the end of every one-minute period, and then down-sampled to the five-minute frequency (we describe the procedure to calculate option returns in Section 4.4). Finally, the daily time-series of the Left Tail Variance (LTV) is obtained from www.tailindex.com.

3.2 Properties of the High-Frequency RNDs and Tail Risk Premium Measure

[Figure 1 about here.]

[14] RV is estimated as in Andersen et al. (2003). IV is estimated as in Mancini (2009). JV is estimated as $\max\{RV - IV, 0\}$.

[15] https://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html

Our tail risk premium measure $\Delta_{\mathbb{Q}}^{\mathbb{P}} ES$ is based on the risk adjustment provided by the risk neutral distribution methodology described in section 2.3.

Figure 1 presents the time series of the physical expected shortfall, $ES^{\mathbb{P}}$, the tail premium $\Delta_{\mathbb{Q}}^{\mathbb{P}} ES$, and the Lagrange multipliers (LMs) for the the S&P500 returns coming from the dual HARA utility problem solved to obtain the RND on each day. A negative LM means the pseudo-investor is long in the index. The third panel in Figure 1 shows that the investor is long in the index throughout our sample. This is a direct consequence of the economic restriction of having a non-negative equity premium that we impose. This restriction also guarantees that our tail risk premium is always above or equal to zero.[16] Note that both $ES^{\mathbb{P}}$ and $\Delta_{\mathbb{Q}}^{\mathbb{P}} ES$ achieve their peaks during the 2008-2009 subprime crisis. Note also that during the financial crisis the pseudo-investor's position in the S&P500 index which arises from the dual portfolio problem defining the RND used to determine the tail risk premium, is smaller than outside of this period. $ES^{\mathbb{P}}$ and the S&P500 realized variance RV are strongly (0.88) correlated. On the other hand, $\Delta_{\mathbb{Q}}^{\mathbb{P}} ES$ has only a correlation of about 0.30 with RV. In what follows, we test the ability of these measures to forecast risk premiums and the future distribution of S&P500 returns.

3.3 Predicting the Equity and Variance Risk premiums

3.3.1 Equity Premium

Table 1 reports the forecasting results for the conditional excess returns on the S&P 500 index (in Panel A) and for gross index returns (in Panel B). Excess returns are obtained over the 3-month risk-free rate reported on Kenneth French's data website, appropriately pro-rated to the short predictive horizon. We report on predictability for the 1-, 5-, and 21-day horizons and several sets of predictors. We first report the predictions based on $\Delta_{\mathbb{Q}}^{\mathbb{P}} ES$

[16] We relax this restriction in Section 4 and compare predictability results. In the unrestricted case, for a significant percent of days the equity premium – measured as the intraday average of the S&P500 returns – is negative. Nonetheless, we do not observe a significant impact of this change on our measure's predictive ability.

and $ES^{\mathbb{P}}$, then we add LTV and RV (as is or decomposed into JV and IV). For the excess returns regressions we add the VRP since it has been shown to robustly forecast the equity risk premium.

For all horizons and predictor sets, only $\Delta_{\mathbb{Q}}^{\mathbb{P}}ES$ is significant at 1% for the 1-day horizon with VRP being significant at 5%. The adjusted R^2 for the set $\{\Delta_{\mathbb{Q}}^{\mathbb{P}}ES, ES^{\mathbb{P}}, LTV, RV\}$ is sizable at 1.3% for gross returns and rises to 2% for excess returns, in a multivariate regression where VRP, significant at a 5% confidence level, plays a complementary role to $\Delta_{\mathbb{Q}}^{\mathbb{P}}ES$. This level of predictability can be rationalized by the fact that $\Delta_{\mathbb{Q}}^{\mathbb{P}}ES$ captures the component of the equity risk premium associated with the tail of the distribution, confirming in a high-frequency environment that aversion to downside risk is an important determinant of the market equity premium.

In a recent paper, Andersen et al. (2020) also find significant predictive power of the tail risk premium for future returns of international stock indices at the 4-6 month horizon using daily observations of option indices. In their study, similarly to our findings, market volatility risk is irrelevant for forecasting the equity premium.

[Table 1 about here.]

3.3.2 Variance Risk Premium

Sections 2.3 and 2.4 suggest that the proposed tail risk premium measure $\Delta_{\mathbb{Q}}^{\mathbb{P}}ES$ captures the variation in investors' aversion to downside risk that should be compensated with a risk premium. In Section 3.3.1, we show that $\Delta_{\mathbb{Q}}^{\mathbb{P}}ES$ robustly forecasts the one-day ahead equity premium, even when we control for other (tail) risk and risk premium measures. In this section, we move a step forward and use predictive regressions to verify the ability of $\Delta_{\mathbb{Q}}^{\mathbb{P}}ES$ to forecast the variance risk premium (VRP). Bollerslev and Todorov (2011) demonstrate that both the equity and variance premiums are to a large extent determined by tail risk compensation. Table 2 shows that our tail risk premium measure strongly predicts the VRP for all the horizons analyzed (1-, 5- and 21-day) with R^2's ranging from 0.23 to 0.32

in multivariate regressions that include the most relevant benchmarks.[17] The benchmarks adopted are RV, LTV, $ES^{\mathbb{P}}$, and the lagged VRP. In addition to the lagged VRP, $\Delta_{\mathbb{Q}}^{\mathbb{P}}ES$ appears as the unique statistically significant regressor. Moreover, both appear to be relevant to forecast the VRP having a similar contribution to the R^2's.

[Table 2 about here.]

The two remaining panels of the Table report, respectively, the patterns of predictability of RV and VIX as dependent variables. Lagged RV is of course a good predictor of next day variance but $ES^{\mathbb{P}}$, LTV, and $\Delta_{\mathbb{Q}}^{\mathbb{P}}ES$ are also significant at the 1% level. For the 5-day and 21-day horizon, RV keeps forecasting power but $\Delta_{\mathbb{Q}}^{\mathbb{P}}ES$, LTV, and $ES^{\mathbb{P}}$ are the strongest predictors.[18] The R^2 for the various specifications are of the order of 60-70%. For the VIX, R^2's are higher with $ES^{\mathbb{P}}$, $\Delta_{\mathbb{Q}}^{\mathbb{P}}ES$ and lagged VRP explaining up to 89% of the variability, and with a complete model that also includes lagged RV having R^2's close to 95%.

Our results confirm those of Bollerslev et al. (2015) and Andersen et al. (2015) in a high-frequency environment and without the use of options data: the existence of a measure of market perception of tail risk compensation is important to predict the equity and variance risk premiums.

3.3.3 The Cross section of Characteristic-sorted Portfolio Returns

We conclude our predictive analysis with a complementary forecasting exercise of the premiums on several sets of portfolios sorted according to the oft-used characteristics of size, book-to-market, profitability, investment, momentum, reversal, and industry. These characteristics form the basis of the factor models of Fama and French (1993, 2015) and Hou et al. (2014).

[17] We measure the daily VRP by $RV \times 365 - (\frac{VIX}{100})^2$ emphasizing the short-term component of the variance premium. This is fully compatible with the way we calculate our tail risk premium measure.

[18] When we decompose RV into IV and JV the continuous component IV performs as RV while the JV gains some forecasting power at the 21-day horizon, while LTV and the two shortfall-based predictors keep their significance.

We report in Figure 2 the results of our 1-day ahead predictive regressions of the characteristic-based portfolios in a bar-chart format. For each characteristic, we forecast the daily returns of the set of ten decile portfolios with four sets of high-frequency predictors: i) $\Delta_{\mathbb{Q}}^{\mathbb{P}}ES$, ii) $\Delta_{\mathbb{Q}}^{\mathbb{P}}ES$ and $ES^{\mathbb{P}}$, iii) $\Delta_{\mathbb{Q}}^{\mathbb{P}}ES$ and RV, iv)$\Delta_{\mathbb{Q}}^{\mathbb{P}}ES$ and LTV. For each decile portfolio we plot a corresponding group of four t-statistics on $\Delta_{\mathbb{Q}}^{\mathbb{P}}ES$ calculated with Andrews (1991) standard errors.

For all sets of predictors and all characteristics, the large majority of t-statistics on $\Delta_{\mathbb{Q}}^{\mathbb{P}}ES$ reported in Panel (a) of Figure 2 are between 2 and 4. This finding, coupled with the individual non-significance of $ES^{\mathbb{P}}$, RV, and LTV reported in Panel (a) of Figure 3, puts forward that $\Delta_{\mathbb{Q}}^{\mathbb{P}}ES$ is the main predictor of the portfolio equity premiums, which extends our central result on the predictability of the market premium in Table 1. In Panel (b) of Figure 2 almost all the R^2 are between 0.5 and 1.3%, which is of the same order of magnitude obtained for the market returns in Table 1. The very low R^2 in Panel (b) of Figure 3 also confirms that the main part of the predictive power obtained for all characteristic-based factors and market returns draws its origin from the tail risk premium $\Delta_{\mathbb{Q}}^{\mathbb{P}}ES$.

[Figure 2 about here.]

[Figure 3 about here.]

3.4 Predicting the Distribution of S&P 500 Returns

Our analysis of the predictive power of $\Delta_{\mathbb{Q}}^{\mathbb{P}}ES$ and $ES^{\mathbb{P}}$ for risk premiums demonstrates that it is primarily the tail premium rather than the level of tail risk that contains relevant predictive information for both equity and variance risk premiums. In this section, we turn to an analysis of the predictive power of $\Delta_{\mathbb{Q}}^{\mathbb{P}}ES$ and $ES^{\mathbb{P}}$ for the distribution of future S&P 500 returns. This analysis might help to explain the mechanisms of variation in the premium that is associated with the variation in $\Delta_{\mathbb{Q}}^{\mathbb{P}}ES$ and also determine why risk itself has little predictive power for the equity premium.

3.4.1 Quantiles of the Distribution of S&P 500 Returns

In the preceding sections we put forward that our nonparametric high-frequency measure of the tail risk premium $\Delta^{\mathbb{P}}_{\mathbb{Q}}ES$ is a significant predictor of the equity and variance risk premiums at short horizons. We also included important control variables as predictors ($ES^{\mathbb{P}}$, RV, VRP and LTV) identifying that some play a role in the prediction of the above-mentioned premiums.

To better understand how these variables affect the distribution of future S&P500 returns, we adopt the conditional quantile regression framework introduced by Koenker and Gilbert (1978). The conditional quantile model for S&P500 daily returns $\{r_t\}_{t=1,\ldots,T_q}$, reads as follows:

$$Q_{r_{t+h}}(\tau|p_t) = \theta_0(\tau) + \theta_1(\tau)\Delta^{\mathbb{P}}_{\mathbb{Q}}ES_t + \theta_2(\tau)ES_t + \theta_3(\tau)LTV_t + \theta_4(\tau)RV_t + \theta_5(\tau)VRP_t \quad (16)$$

where $p_t = \{1, \Delta^{\mathbb{P}}_{\mathbb{Q}}ES_t, ES^{\mathbb{P}}_t, LTV_t, RV_t, VRP_t\}$ and the θ_js are functions mapping $\tau \in [0,1]$ into $\mathbb{R}$. This equation says that the conditional τ-quantile of the daily S&P 500 return distribution at time $t+h$ is a linear function of the tail risk and variance measures at time t. In the quantile regression of r_{t+h} on the variables in p_t, the regression coefficients θ_τ are estimated by minimizing the quantile-weighted absolute value of errors:

$$\hat{\theta}_\tau = \arg\min_{\theta_\tau \in \mathbb{R}^5} \sum_{t=1}^{T_q-h} (\tau \cdot \mathbf{1}(r_{t+h} \geq p_t\theta_\tau)|r_{t+h} - p_t\theta_\tau| + (1-\tau) \cdot \mathbf{1}(r_{t+h} < p_t\theta_\tau)|r_{t+h} - p_t\theta_\tau|) \quad (17)$$

where $\mathbf{1}(.)$ is the indicator function. The predicted quantile conditional on p_t is:

$$\hat{Q}_{r_{t+h}}(\tau|p_t) = p_t\hat{\theta}_\tau \quad (18)$$

We estimate predictive quantile regressions for the one-day-ahead distribution of S&P 500 index returns over the full sample from January 2, 2004 to December 31, 2018. The regressions are estimated for the 5th through the 95th percentiles in 10-percentage point increments, and for the median. We report the estimated coefficients and their standard deviations for the 1-day ($h = 1$) horizon for all quantiles in Table 3.

We first examine the differential contribution of $\Delta_{\mathbb{Q}}^{\mathbb{P}} ES$ and $ES^{\mathbb{P}}$ to the distribution of index returns. The estimated coefficients clearly indicate that an increase in risk ($ES^{\mathbb{P}}$) leads to a larger probability of observing extreme negative and positive returns, whereas an increase in the tail premium ($\Delta_{\mathbb{Q}}^{\mathbb{P}} ES$) shifts the quantiles around the median, and the whole right tail, towards more positive values, with a stronger effect for right-tail quantiles, and no significant effect on the left tail.[19]

Among the other variables in the quantile model, the impact of the VRP on the distribution of index returns is the most interesting. Recall that it is the VRP rather than RV (level of risk) that has been documented to predict future returns – a more negative VRP is associated with higher future returns. Its impact on the return distribution is clearly stronger than that of RV, and similar to, yet weaker than, that of $ES^{\mathbb{P}}$. The switched signs of coefficient with respect to $ES^{\mathbb{P}}$ simply reflect that when VRP is more negative the variance risk premium is larger, and larger VRP leads to larger probability of observing extreme returns, specially positive ones.

Now that we have identified which regions of the return distributions each of our variables affect, to evaluate the quality of the estimated conditional distribution, we need to conduct a thorough out-of-sample study of conditional interval forecasts. This is the topic of the next section.

[Table 3 about here.]

[Table 4 about here.]

[19]The coefficients of the tail risk premium $\Delta_{\mathbb{Q}}^{\mathbb{P}} ES$ are significant at the 0.01 level for all quantiles between 0.35 and 0.95.

3.4.2 Out-of-sample forecasts of the S&P 500 return distribution

To assess the actual predictive out-of-sample power of our measures for various parts of the return distribution, we rely on the framework developed by Christoffersen (1998) to evaluate conditional interval forecasts. Namely, we consider the likelihood ratio tests of interval forecast conditional coverage (CC), which are comprised of the joint tests of interval forecast error independence (ID) and unconditional coverage (UC).

For model evaluation, we split our data into the estimation sample which contains 75% of the data (2,820 observations starting on January 2nd, 2004 and ending on April 7th, 2015), and the evaluation sample (the remaining 940 observations ending on December 31st, 2018).

The results are reported in Table 4. For all intervals as defined previously, we report the p-values of the CC, ID and UC tests for several conditioning tests based on several individual and groups of predictors. The predictor sets are as follows: i) $ES^{\mathbb{P}}$, ii) $\Delta^{\mathbb{P}}_{\mathbb{Q}}ES$, iii) $ES^{\mathbb{P}}$ and $\Delta^{\mathbb{P}}_{\mathbb{Q}}ES$, iv) VRP and LTV, v) VRP, LTV, and $ES^{\mathbb{P}}$, vi) VRP, LTV, and $\Delta^{\mathbb{P}}_{\mathbb{Q}}ES$, and finally vii) all regressors.[20] We also conduct a joint evaluation of interval forecasts that aggregates forecasts to six intervals spanning the quantiles (0.0 to 0.05], (0.05 to 0.25], (0.25 to 0.5], (0.5 to 0.75], (0.75 to 0.95], and (0.95 to 1.00].

For $ES^{\mathbb{P}}$, the null hypothesis of correct conditional coverage is rejected only for the [0.25,0.35] interval, while the independence of prediction errors is never rejected. For the joint interval test, CC and UC are rejected. When we add $\Delta^{\mathbb{P}}_{\mathbb{Q}}ES$ to $ES^{\mathbb{P}}$ in the column titled Both, results clearly improve with only UC rejected at the 5% level in the joint test, yet no rejection in the CC test. For the various intervals the results are very supportive of interval forecast efficiency for all three tests except for one interval in each individual test. Moreover, the inclusion of $\Delta^{\mathbb{P}}_{\mathbb{Q}}ES$ in any subset of regressors improves the out-of-sample fit for the middle of the S&P 500 returns distribution confirming that our measure helps to forecast expected returns by better identifying the center of the distribution. The best

[20]Due to the fact that VRP has a much better performance than RV in the previous quantile regressions we decided to drop RV. Results including RV are qualitatively similar and available upon request.

conditioning sets are $ES^{\mathbb{P}} + \Delta^{\mathbb{P}}_{\mathbb{Q}}ES$ and $VRP + LTV + ES^{\mathbb{P}} + \Delta^{\mathbb{P}}_{\mathbb{Q}}ES$. The former misses only the 25th–35th percentile while the latter does not perform well for the 95th–100th percentile. In joint evaluations, both fail only the unconditional coverage test, albeit the null of correct conditional coverage cannot be rejected. Overall, the results obtained with the quantile model support that our tail risk premium measure $\Delta^{\mathbb{P}}_{\mathbb{Q}}ES$ is fundamental in determining the future distribution of S&P 500 returns.

4 Robustness Analysis

Our tail risk measure depends on a number of choices regarding the estimator for the risk-neutral distributions. In this section, we perturb the method to learn about its sensitivity and compare its performance to the benchmark case whose empirical properties were analyzed above.

Our benchmark measure relies on a particular value of the parameter that indexes the Cressie-Read family ($\gamma = -3$) and adopts only S&P 500 high-frequency returns. Another design choice is to impose a non-negative risk premium for the S&P 500 to prevent the pseudo marginal investor solving the Cressie Read dual portfolio problem from shorting the index in bad times. By doing so, we limit the tilting towards extreme negative returns but by the same token we temper the importance of tail events in our risk-neutral expected shortfall measurement. We will therefore do away with this restriction to check how it affects our empirical findings.

As we explained it in Section 2.3, the value of $\gamma = -3$ was chosen to tilt the distribution of returns towards the more negative returns, as an investor with higher aversion to downside risk will do. In this section, we check if this choice is essential to our findings or if a less drastic risk-adjustment will produce similar results. We also verify if including options in the estimation of the risk neutral measure improve the ability of our tail risk measure to predict equity and variance risk premiums.

Finally, our measure can be computed for any index or security where high-frequency returns are available. In this paper, we chose to compute the risk-neutral expected shortfall with a nonparametric RND extracted from high-frequency index returns only. However, it is natural to ask whether adding option returns to index returns to extract the nonparametric RND, as explained in section 2.3, helps better predict the equity premium measures of the S&P 500 index. We use high-frequency data on S&P 500 options to test this possibility.

4.1 Risk Neutralization with the Hellinger Discrepancy

The Hellinger discrepancy is a member of the Cressie-Read family of discrepancy measures and is obtained for the value of -0.5 of the indexing parameter γ. In Appendix A, we reproduce the main premiums predictability results with the new $\Delta_{\mathbb{Q}}^{\mathbb{P}} ES$ obtained with the use of the Hellinger measure (Table A.1 and Figure A.1). As in the benchmark case, $\Delta_{\mathbb{Q}}^{\mathbb{P}} ES$ appears as the main predictor for the equity and variance risk premiums confirming that our findings are robust to a change of dispersion measure within the Cressie Read family.

4.2 Risk Neutralization Without Restrictions on the Equity Premium

Recall that when we recover the risk-neutral probability measure for the purpose of our main analysis (using the optimization problem (6)), we impose a positive risk premium on days with negative realized returns (i.e., a negative average of returns implying a negative equity premium). In this section we drop this restriction and after re-calculating $\Delta_{\mathbb{Q}}^{\mathbb{P}} ES$ we conclude that this has no material impact on our predictive results. The supplementary results are collected in Table B.1 and in Figure B.1 in Appendix B.

The immediate consequence of allowing for a negative risk premium is that the estimates of the Lagrange multiplier λ turn positive on a significant part of the sample, as evident in the rightmost panel of Figure B.1. This, in turn, changes our estimates of $\Delta_{\mathbb{Q}}^{\mathbb{P}} ES$, because for $\lambda > 0$ the pseudo investor solving the Cressie Read dual portfolio problem sells the index making the implied RND to put higher probability weights in states of nature where the index has large positive returns. As a consequence, ES becomes smaller under Q than under P, as visible in the central panel of Figure B.1. This difference notwithstanding, a visual inspection of the central panels of Figures B.1 and our main Figure 1 uncovers similar dynamics of $\Delta_{\mathbb{Q}}^{\mathbb{P}} ES$ with and without restrictions. A subsequent comparison of the predictive power of $\Delta_{\mathbb{Q}}^{\mathbb{P}} ES$ for the expected excess return on the S&P 500 index (see Tables B.1 and 1) suggests that removing the positive equity premium constraint renders the estimated $\Delta_{\mathbb{Q}}^{\mathbb{P}} ES$ more noisy (the R^2s and t-statistics of 1-day horizon regressions become lower). On the other hand, the unrestricted estimate of $\Delta_{\mathbb{Q}}^{\mathbb{P}} ES$ becomes a significant predictor of five-day returns once controls are added to the regressions. Beyond the first moment, we do not observe substantial differences in the predictive regressions of the VRP.

4.3 Lower Expected Shortfall Threshold

In our final robustness exercise we examine the sensitivity of our results to the choice of the expected shortfall threshold s_α defined in equations (3) and (4). While in our core analysis we set $\alpha = 0.2$, in this section we present the results for $\alpha = 0.1$. The results are collected in Table C.1 and Figure C.1 in Appendix C.

First of all, the change in the choice of α mechanically translates to an increase in the ES measure, which is evident when comparing the left-most panels of Figures C.1 and 1. Somewhat less intuitively, we also observe changes to the estimates of the tail risk premium, $\Delta_{\mathbb{Q}}^{\mathbb{P}} ES$, comparing the central panels of the aforementioned figures. With $\alpha = 0.1$, the tail risk premium estimates become significantly higher than with $\alpha = 0.2$ in periods of market distress. This is compatible with heterogeneous investor's attitude towards downside risk observed under different extreme quantiles of the distribution of returns[21].

With the new threshold for expected shortfall calculation, we do not observe any changes

[21] See Castro and Galvao (2019) for the development of a rational dynamic model based on quantile utility preferences.

for the predictive power of $ES^{\mathbb{P}}$ and $\Delta_{\mathbb{Q}}^{\mathbb{P}}ES$ for S&P 500 equity and variance risk premiums. Most importantly, $ES^{\mathbb{P}}$ still fails to predict the return on the index in univariate regressions in Table C.1.

4.4 Including Option Returns in the Estimation of the Risk Neutral Measure

We apply the following filters to the option data. For each day, we use the options with maturity closest to one month, taking the shorter maturity as a tie breaker. We remove all quotes with zero bid prices and those where the ask price is more than five times the bid price. The returns on index options are calculated as follows. In each five-minute window, we sort out of the money (OTM) call (put) options into five portfolios based on the absolute value of their Black-Scholes delta, denoted as $\Delta 01$ (deep OTM options with absolute deltas ranging from 0.0 to 0.1) through $\Delta 05$ (close to at the money (ATM) options with absolute deltas ranging from 0.4 to 0.5). Next, we calculate the mid price for each option. In the following step, we match the observations in a given five-minute window to those in the subsequent window, and we discard options with no match. We further drop from the return calculation the options whose mid prices did not change between two observation windows. Finally, from the remaining data we calculate the equally-weighted return on each option portfolio.

We estimate the RN measure by solving the minimum-discrepancy problem in (6) including as basis assets the S&P 500 returns and the returns of one group of options characterized by their delta (Δ). In Table 5, the columns correspond to option returns grouped by Δ's. As mentioned before, $\Delta 01$ corresponds to deep out-of-the-money options while $\Delta 05$ corresponds to at-the-money options. We include only $\Delta_{\mathbb{Q}}^{\mathbb{P}}ES$ and $ES^{\mathbb{P}}$ as predictors and consider the 1-day horizon. The tail risk premium $\Delta_{\mathbb{Q}}^{\mathbb{P}}ES$ is significant for the call groups Call $\Delta 01$ and Call $\Delta 03$ groups. In terms of R^2 this model does worse than the equivalent one without options in Table 1 (0.0069 and 0.0065 for Call $\Delta 01$ and Call $\Delta 03$, respectively, against 0.0075). These results could be rationalized by the large noise associated with high-frequency returns on options.[22]

[Table 5 about here.]

5 Conclusion

In this paper, we propose a new method to measure tail risk premium in high frequency markets using minimal information on an index of stocks. The method uses solely intra-day index returns and a risk neutralization algorithm. We theoretically show that the tail risk premium, defined as a difference of expected shortfalls under the risk neutral and physical probabilities, should be sensitive to the shape of the tail of index returns and therefore able to predict future returns and premiums (Bollerslev et al. (2015)). We empirically test the ability of our tail risk premium measure to predict equity and variance risk premiums of the S&P500 returns. We show that it robustly predicts risk premiums not only of the S&P 500 returns but also of characteristic-based portfolio returns. A natural extension of our method would be to use a cross section of high frequency returns to identify if and by how much bringing cross-sectional information improves the model forecasting ability.

[22]When we include option and index returns together in the estimation of the RN measure, we lose the interpretation of how the RN measure acts on index returns. When the S&P500 is the unique asset, the RN measure over-weights its negative returns and under-weights positive ones, provided that we restrict the equity premium to be positive. In contrast, when the index and an option are basis assets, the RN measure now over-weights negative returns coming from the optimal solution of the dual portfolio problem, i.e., a linear combination of index and option returns.

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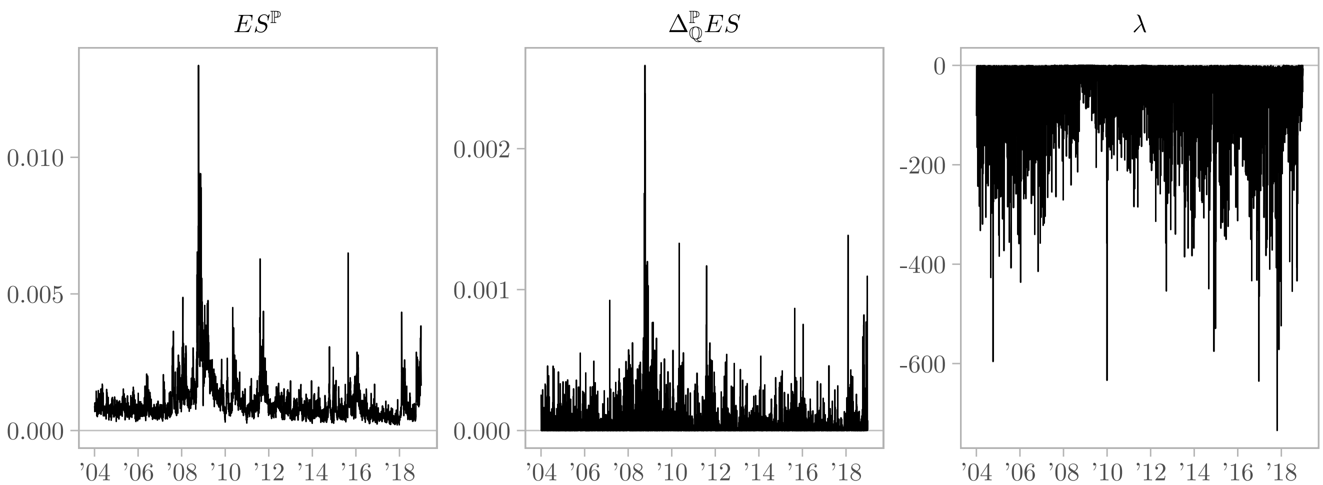
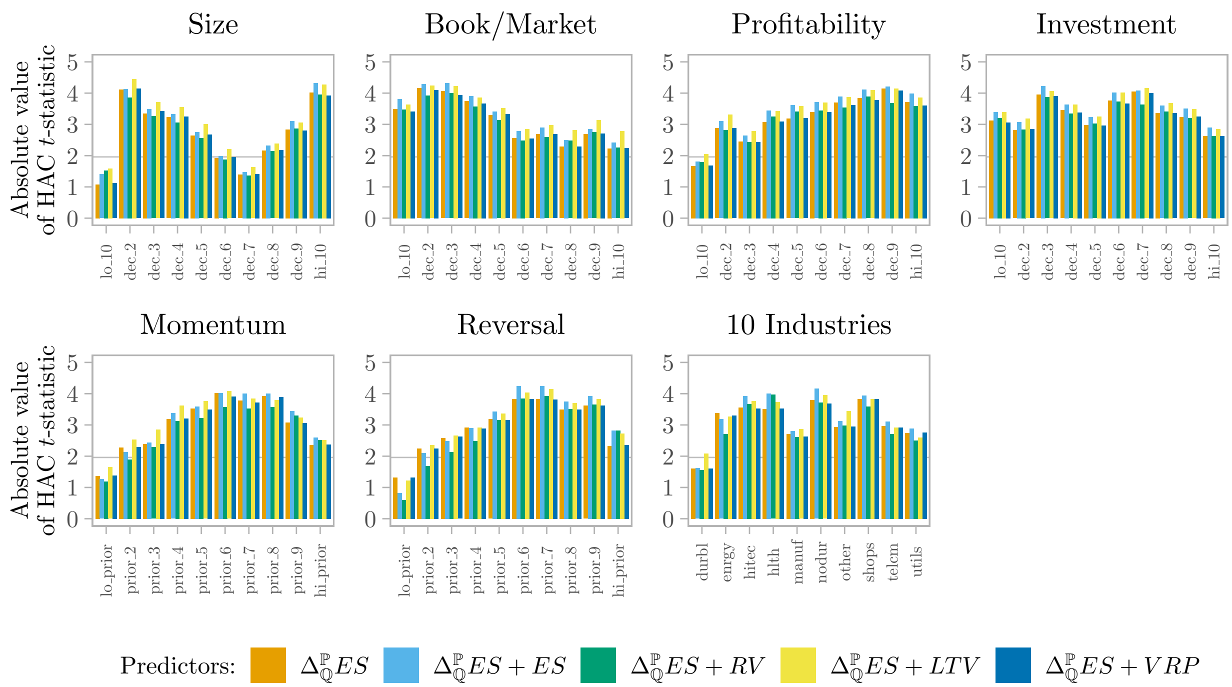


Figure 1: Time series of the tail measures implied by the S&P 500 intraday data.

Figure 2: Predictive regressions: Fama-French portfolios

Summary of predictive regressions of the 1-day ahead excess return on Fama-French portfolios of stocks sorted on Size, Book/Market, Profitability, Investment, Momentum, Reversal, and the 10 Industry portfolios. Four regression specifications are considered, using the following regressors: (i) $\Delta^{\mathbb{P}}_{\mathbb{Q}}ES$, (ii) $\Delta^{\mathbb{P}}_{\mathbb{Q}}ES$ and $ES^{\mathbb{P}}$, (iii) $\Delta^{\mathbb{P}}_{\mathbb{Q}}ES$ and RV, (iv) $\Delta^{\mathbb{P}}_{\mathbb{Q}}ES$ and LTV. We report Andrews (1991) standard errors calculated with the use of the `sandwich 3.0.0` package for `R 4.0.3` (Zeileis et al., 2020).

(a) t-statistics of $\Delta^{\mathbb{P}}_{\mathbb{Q}}ES$ in predictive regressions



(b) R^2 of predictive regressions

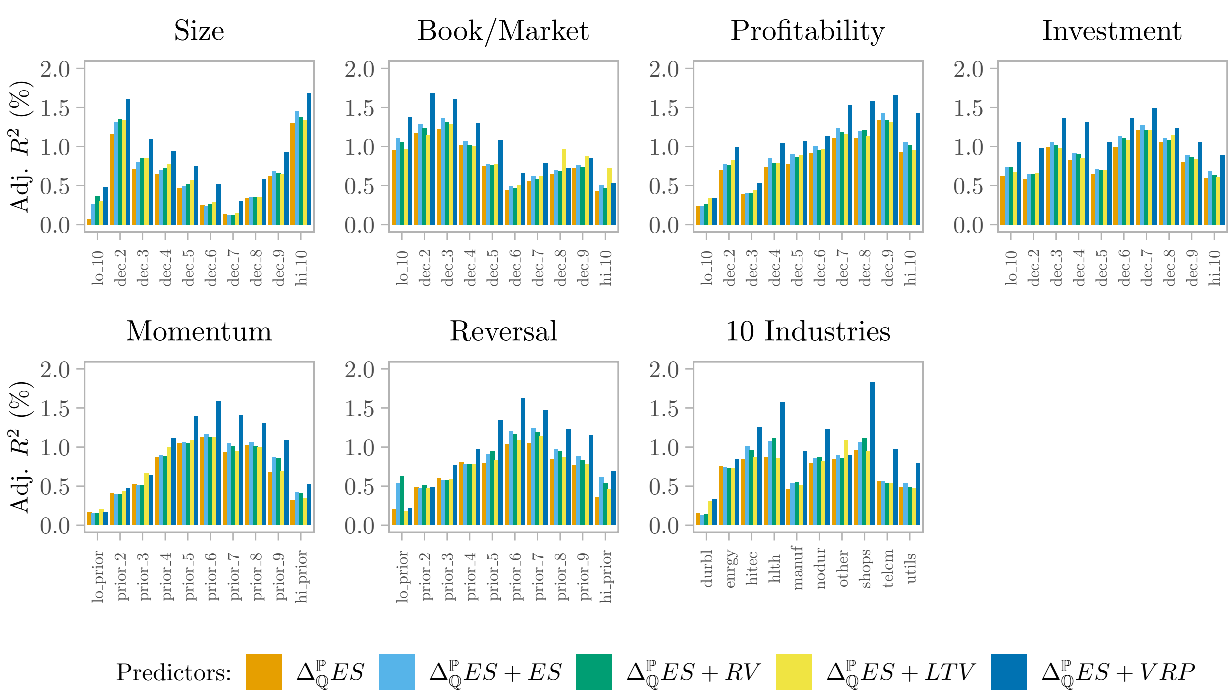
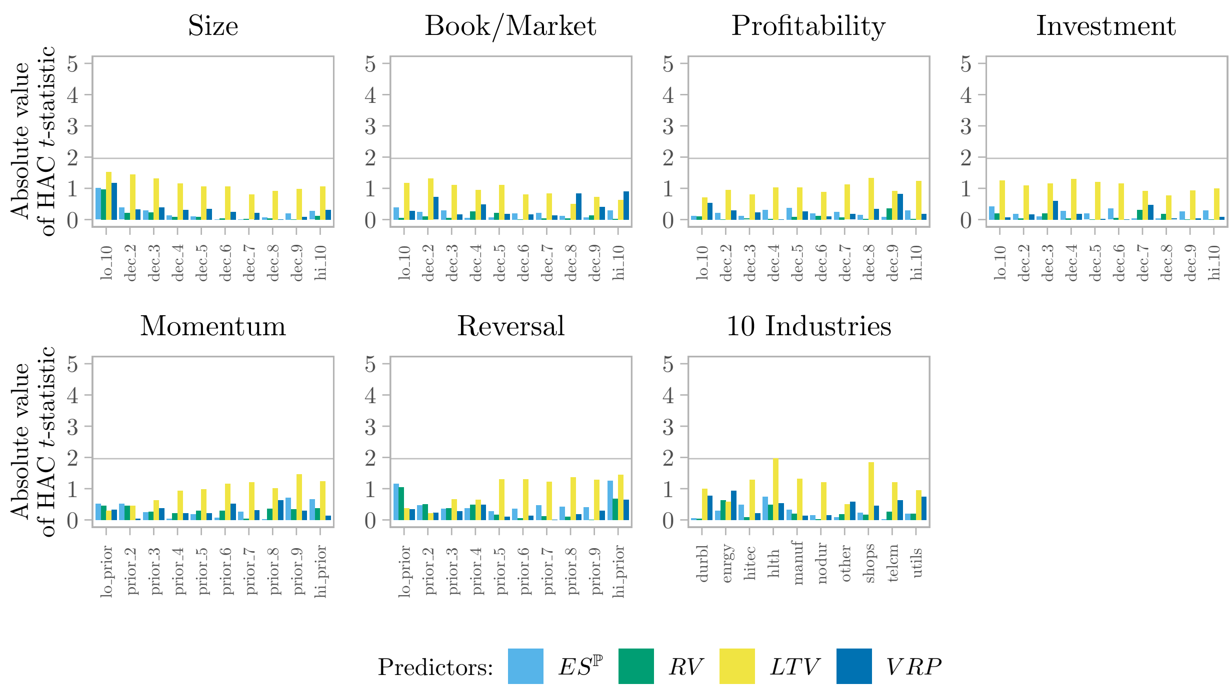


Figure 3: Predictive regressions: Fama-French portfolios, alternative predictors

Summary of predictive regressions of the 1-day ahead excess return on Fama-French portfolios of stocks sorted on Size, Book/Market, Profitability, Investment, Momentum, Reversal, and the 10 Industry portfolios. Four regression specifications are considered, using the following regressors: (i) $ES^{\mathbb{P}}$, (ii) RV, (iii) LTV. We report Andrews (1991) standard errors calculated with the use of the `sandwich 3.0.0` package for `R 4.0.3` (Zeileis et al., 2020). In order to facilitate the comparison with the main predictability results for Fama-French portfolios, the axis scales are set the same as in Figure 2.

(a) t-statistics of alternative predictors



(b) R^2 of predictive regressions

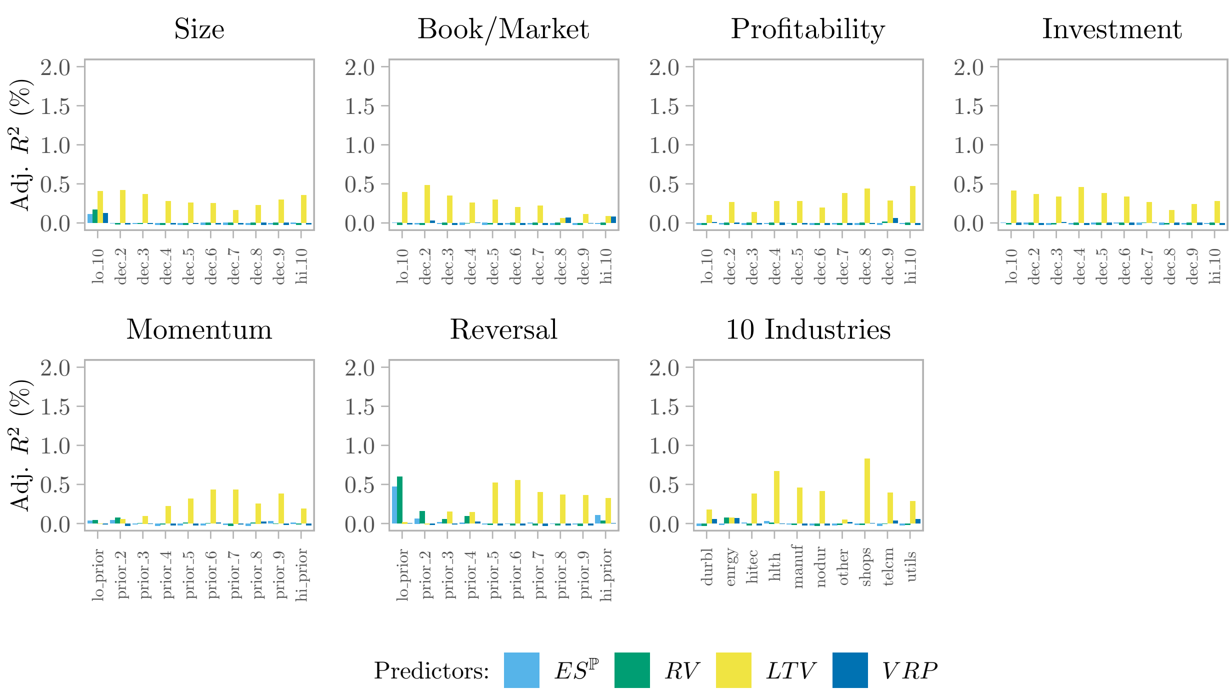


Table 1: Predictive regressions: S&P 500 returns

The table reports regression coefficients and their standard deviations (in parentheses) of predictive regressions of the close-to-close S&P 500 returns at the 1-, 5- and 21-day horizons. The coefficients that are significant at the 0.05 confidence level are printed in bold and those significant at the 0.01 level are additionally highlighted with a $\star$. The 5- and 21-day horizon regressions are estimated on overlapping observations using the approach of Britten-Jones et al. (2011). We report Andrews (1991) standard errors calculated with the use of the `sandwich 3.0.0` package for `R 4.0.3` (Zeileis et al., 2020). $ES^{\mathbb{P}}$ is the non-parametric estimate of realized (physical) Expected Shortfall of intra-day S&P 500 returns before the close, on the day when each return calculation commences, $ES_t^{\mathbb{P}} = -E_t[R_{it}|R_{it} \leq F_{R_t}^{-1}(0.2)]$. Our Tail Risk Premium measure, $\Delta_{\mathbb{Q}}^{\mathbb{P}} ES$, is the difference between the risk-neutral and the physical ES. LTV is the Left Tail Variance of Bollerslev et al. (2015) obtained from www.tailindex.com. RV is the realized variance of intra-day returns on the S&P500 Index. IV is an estimate of integrated quadratic variation, the continuous component of realized variance, estimated as in Mancini and Gobbi (2012). JV is the jump component of realized variance, calculated as $\max\{RV - IV, 0\}$. RV, IV, and JV are calculated from return data sampled at the 5-minute frequency.

Panel A: Equity Risk Premium (Excess Return on the S&P 500 Index)

	1 days	5 days	21 days	1 days	5 days	21 days	1 days	5 days	21 days	1 days	5 days	21 days	1 days	5 days	21 days
$ES^{\mathbb{P}}$	0.17	0.51	−1.67	−0.16	0.03	−2.14	−2.04	−2.61	−5.10	−2.01	−2.63	−4.99			
	(0.77)	(2.88)	(10.08)	(0.87)	(2.77)	(9.47)	(1.15)	(4.48)	(13.96)	(1.13)	(4.41)	(13.88)			
$\Delta_{\mathbb{Q}}^{\mathbb{P}} ES$				**7.50***	11.17	11.33	**8.83***	8.82	3.80	**8.75***	8.81	6.72	**7.19***	11.29	9.15
				(2.90)	(8.33)	(19.69)	(2.83)	(7.65)	(18.27)	(2.66)	(7.25)	(18.28)	(2.68)	(10.60)	(30.49)
LTV							−0.10	−0.24	−0.48	−0.10	−0.24	−0.49			
							(0.07)	(0.27)	(0.70)	(0.07)	(0.27)	(0.70)			
RV							15.25	28.22	50.52						
							(10.36)	(39.78)	(122.31)						
VRP							**−15.90**	−26.59	−82.45	**−16.05**	−27.06	−80.96	1.53	−1.95	−27.47
							(7.74)	(25.59)	(82.05)	(7.79)	(25.61)	(83.73)	(9.07)	(17.68)	(42.73)
JV										17.73	29.57	1.14			
										(14.14)	(45.40)	(150.93)			
IV										14.79	28.25	54.19			
										(10.81)	(40.45)	(121.18)			
Constant	0.00	0.00	0.01	−0.00	0.00	0.01	0.00	0.00	0.01	0.00	0.00	0.01	−0.00	0.00	0.00
	(0.00)	(0.00)	(0.01)	(0.00)	(0.00)	(0.01)	(0.00)	(0.00)	(0.01)	(0.00)	(0.00)	(0.01)	(0.00)	(0.00)	(0.00)
R^2 (adj)	−2.4e-06	−9.4e-05	−9e-05	0.0073	0.0032	0.00088	0.02	0.0051	0.006	0.02	0.005	0.0058	0.0075	0.0027	0.0011

Panel B: Return on the S&P 500 Index

	1 days	5 days	21 days	1 days	5 days	21 days	1 days	5 days	21 days	1 days	5 days	21 days
$ES^{\mathbb{P}}$	0.16	0.49	−1.81	−0.16	−0.00	−2.30	−0.79	−0.48	1.67	−0.76	−0.49	1.59
	(0.77)	(2.88)	(10.08)	(0.87)	(2.77)	(9.47)	(1.08)	(3.71)	(11.24)	(1.06)	(3.73)	(11.01)
$\Delta_{\mathbb{Q}}^{\mathbb{P}} ES$				**7.51***	11.23	11.52	**10.60***	11.87	14.31	**10.62***	12.06	17.52
				(2.90)	(8.34)	(19.70)	(2.92)	(6.80)	(13.80)	(2.74)	(6.68)	(13.28)
LTV							−0.01	−0.09	−0.02	−0.01	−0.09	−0.04
							(0.07)	(0.21)	(0.69)	(0.08)	(0.21)	(0.67)
RV							1.18	4.81	−22.49			
							(8.95)	(24.42)	(70.14)			
JV										1.12	1.53	−83.95
										(13.31)	(36.72)	(101.20)
IV										1.02	5.18	−15.13
										(9.84)	(26.38)	(67.94)
Constant	0.00	0.00	0.01	0.00	0.00	0.01	0.00	0.00	0.01	0.00	0.00	0.01
	(0.00)	(0.00)	(0.01)	(0.00)	(0.00)	(0.01)	(0.00)	(0.00)	(0.01)	(0.00)	(0.00)	(0.01)
R^2 (adj)	0.00017	7.5e-05	9.9e-05	0.0075	0.0034	0.0011	0.013	0.003	0.00093	0.013	0.0028	0.0019

Table 2: Predictive regressions: Variance Risk Premium and its Components

The table reports regression coefficients and their standard deviations (in parentheses) for predictive regressions of the Variance Risk Premium and its components: realized variance, measured from the intra-day S&P 500 returns, and the variance swap rate, measured with the use of the VIX index, at the 1-, 5- and 21-day horizons. The coefficients that are significant at the 0.05 confidence level are printed in bold and those significant at the 0.01 level are additionally highlighted with a $\star$. The 5- and 21-day horizon regressions are estimated on overlapping observations using the approach of Britten-Jones et al. (2011). We report Andrews (1991) standard errors calculated with the use of the `sandwich 3.0.0` package for `R 4.0.3` (Zeileis et al., 2020). $ES^{\mathbb{P}}$ is the non-parametric estimate of realized (physical) Expected Shortfall of intra-day S&P 500 returns before the close, on the day when each return calculation commences, $ES_t^{\mathbb{P}} = -E_t[R_{it}|R_{it} \leq F_{R_t}^{-1}(0.2)]$. Our Tail Risk Premium measure, $\Delta_{\mathbb{Q}}^{\mathbb{P}} ES$, is the difference between the risk-neutral and the physical ES. LTV is the Left Tail Variance of Bollerslev et al. (2015) obtained from `www.tailindex.com`. RV is the realized variance of intra-day returns on the S&P500 Index. IV is an estimate of integrated quadratic variation, the continuous component of realized variance, estimated as in Mancini and Gobbi (2012). JV is the jump component of realized variance, calculated as $\max\{RV - IV, 0\}$.

	1 days	5 days	21 days	1 days	5 days	21 days	1 days	5 days	21 days	1 days	5 days	21 days
Panel A: Variance Risk Premium												
$ES^{\mathbb{P}}$	0.02	0.05	−0.16	−0.00	0.01	−0.22	**−0.02⋆**	**−0.10**	**−0.57⋆**	−0.03	−0.07	−0.20
	(0.01)	(0.07)	(0.24)	(0.01)	(0.05)	(0.17)	(0.01)	(0.04)	(0.17)	(0.02)	(0.06)	(0.23)
$\Delta_{\mathbb{Q}}^{\mathbb{P}} ES$				**0.38⋆**	**0.84⋆**	**1.43⋆**	**0.40⋆**	**0.93⋆**	**1.75⋆**	**0.22⋆**	**0.53⋆**	**1.36⋆**
				(0.14)	(0.27)	(0.51)	(0.14)	(0.23)	(0.53)	(0.06)	(0.12)	(0.32)
VRP							**0.32⋆**	**1.59⋆**	**5.13⋆**	**0.38⋆**	**1.69⋆**	**6.40⋆**
							(0.07)	(0.18)	(0.58)	(0.10)	(0.43)	(1.21)
LTV										0.00	0.00	−0.01
										(0.00)	(0.00)	(0.01)
RV										0.01	−0.22	−1.95
										(0.11)	(0.59)	(2.06)
Constant	**−0.00⋆**	**−0.00⋆**	**−0.00⋆**	**−0.00⋆**	**−0.00⋆**	**−0.00⋆**	**−0.00**	**−0.00**	−0.00	−0.00	**−0.00⋆**	**−0.00⋆**
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
R^2 (adj)	0.092	0.087	0.087	0.23	0.23	0.23	0.31	0.32	0.25	0.3	0.32	0.3
Panel B: Realized Variance												
	1 days	5 days	21 days	1 days	5 days	21 days	1 days	5 days	21 days	1 days	5 days	21 days
$ES^{\mathbb{P}}$	**0.17⋆**	**0.79⋆**	**2.79⋆**	**0.14⋆**	**0.72⋆**	**2.58⋆**	0.02	**0.25⋆**	**1.16⋆**	**0.02**	**0.25⋆**	**1.19⋆**
	(0.01)	(0.08)	(0.39)	(0.01)	(0.08)	(0.31)	(0.02)	(0.08)	(0.27)	(0.01)	(0.08)	(0.28)
$\Delta_{\mathbb{Q}}^{\mathbb{P}} ES$				**0.55⋆**	**1.62⋆**	**4.87⋆**	**0.26⋆**	**0.75⋆**	**2.60⋆**	**0.27⋆**	**0.85⋆**	**2.43⋆**
				(0.15)	(0.29)	(0.71)	(0.07)	(0.17)	(0.44)	(0.07)	(0.16)	(0.44)
LTV							**0.00⋆**	**0.02⋆**	**0.05⋆**	**0.00⋆**	**0.02⋆**	**0.05⋆**
							(0.00)	(0.01)	(0.02)	(0.00)	(0.01)	(0.02)
RV							**0.47⋆**	**1.50**	**4.81**			
							(0.08)	(0.59)	(2.22)			
JV										0.24	−0.33	**8.41**
										(0.26)	(1.00)	(3.50)
IV										**0.49⋆**	**1.69**	**4.21**
										(0.07)	(0.71)	(2.08)
Constant	**−0.00⋆**	**−0.00⋆**	**−0.00⋆**	**−0.00⋆**	**−0.00⋆**	**−0.00⋆**	**−0.00⋆**	**−0.00⋆**	**−0.00⋆**	**−0.00⋆**	**−0.00⋆**	**−0.00⋆**
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
R^2 (adj)	0.52	0.54	0.47	0.63	0.61	0.55	0.72	0.68	0.6	0.72	0.68	0.62
Panel C: VIX												
	1 days	5 days	21 days	1 days	5 days	21 days	1 days	5 days	21 days	1 days	5 days	21 days
$ES^{\mathbb{P}}$	**0.15⋆**	**0.74⋆**	**2.95⋆**	**0.15⋆**	**0.71⋆**	**2.81⋆**	**0.17⋆**	**0.84⋆**	**3.25⋆**	0.01	0.10	**0.54**
	(0.01)	(0.06)	(0.23)	(0.01)	(0.06)	(0.20)	(0.01)	(0.06)	(0.21)	(0.01)	(0.07)	(0.23)
$\Delta_{\mathbb{Q}}^{\mathbb{P}} ES$				**0.17⋆**	**0.78⋆**	**3.44⋆**	**0.15⋆**	**0.67⋆**	**3.03⋆**	−0.02	−0.09	0.02
				(0.03)	(0.10)	(0.33)	(0.03)	(0.10)	(0.35)	(0.01)	(0.13)	(0.41)
VRP							**−0.38⋆**	**−1.90⋆**	**−6.64⋆**	**−0.91⋆**	**−4.50⋆**	**−16.50⋆**
							(0.06)	(0.18)	(0.72)	(0.04)	(0.36)	(1.85)
LTV										0.00	0.00	−0.00
										(0.00)	(0.00)	(0.01)
RV										**0.93⋆**	**4.22⋆**	**15.82⋆**
										(0.06)	(0.71)	(2.48)
Constant	**−0.00⋆**	**−0.00⋆**	**−0.00⋆**	**−0.00⋆**	**−0.00⋆**	**−0.00⋆**	**−0.00⋆**	**−0.00⋆**	**−0.00⋆**	−0.00	−0.00	0.00
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
R^2 (adj)	0.8	0.86	0.88	0.81	0.86	0.88	0.87	0.88	0.89	0.96	0.95	0.91

Table 3: Predictive quantile regressions: 1-day S&P 500 returns, full sample

The table reports quantile regression coefficients and their standard deviations (in parentheses) of quantile regressions predicting the distribution of the returns on the S&P 500 index at the 1-day horizon. The sample is from Jan 2nd 2004 to Dec 31 2018. Coefficients which are statistically significant at the 0.05 confidence level are printed in bold. Those significant at the 0.01 level are additionally marked with a $\star$. The standard errors are computed by pairwise bootstrap with the use of the `quantreg 5.83` (Koenker, 2013) package for `R 4.0.3`. The R^1 goodness-of-fit measure is calculated as in Koenker and Machado (1999). $ES^{\mathbb{P}}$ is the non-parametric estimate of realized (physical) Expected Shortfall of intra-day S&P 500 returns before the close, on the day when each return calculation commences, $ES_t^{\mathbb{P}} = -E_t[R_{it}|R_{it} \leq F_{R_t}^{-1}(0.2)]$. Our Tail Risk Premium measure, $\Delta_{\mathbb{Q}}^{\mathbb{P}} ES$, is the difference between the risk-neutral and the physical ES. LTV is the Left Tail Variance of Bollerslev et al. (2015) obtained from `www.tailindex.com`. RV is the realized variance of intra-day returns on the S&P500 Index. The daily value of the VRP is defined as $VRP \equiv RV \times 365 - (VIX/100)^2$.

Quantile:	0.05	0.15	0.25	0.35	0.45	0.5	0.55	0.65	0.75	0.85	0.95
$\Delta_{\mathbb{Q}}^{\mathbb{P}} ES$	0.95	4.57	4.04	**6.00***	**7.65***	**7.75***	**8.72***	**8.14***	**7.25***	**12.12***	**13.29**
	(5.29)	(4.74)	(2.50)	(2.19)	(2.25)	(1.94)	(2.33)	(2.19)	(2.58)	(4.09)	(6.37)
$ES^{\mathbb{P}}$	**−10.66***	**−6.95***	**−4.25***	−1.12	0.12	−0.38	0.43	1.10	**3.05***	**3.78***	**7.41***
	(2.08)	(1.75)	(1.22)	(1.00)	(0.98)	(0.95)	(0.94)	(1.08)	(0.99)	(1.08)	(2.46)
LTV	−0.13	−0.10	−0.05	0.04	−0.00	−0.06	−0.07	−0.10	−0.04	−0.05	−0.11
	(0.13)	(0.14)	(0.09)	(0.08)	(0.07)	(0.06)	(0.06)	(0.06)	(0.08)	(0.07)	(0.16)
RV	−12.88	−8.21	−3.41	−12.52	−6.02	7.03	6.73	18.02	**18.10**	**28.66***	37.41
	(15.37)	(19.08)	(11.31)	(9.24)	(11.71)	(10.28)	(10.70)	(11.29)	(8.94)	(9.45)	(19.34)
VRP	**29.48**	12.77	6.22	2.27	−4.62	**−11.05**	**−13.56**	**−17.74**	**−24.35***	**−38.87***	**−60.51***
	(12.53)	(14.78)	(8.18)	(6.30)	(6.98)	(5.43)	(6.23)	(6.92)	(8.19)	(9.27)	(11.85)
Constant	−0.00	−0.00	0.00	−0.00	−0.00	0.00	0.00	0.00	0.00	**0.00***	0.00
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
R^1	0.22	0.094	0.04	0.015	0.0056	0.0067	0.011	0.029	0.062	0.13	0.3

Table 4: Out-of-sample forecasts of the S&P 500 return distribution: interval tests

The table reports the results of evaluating out-of-sample interval forecasts of the return on the S&P 500 index. Each row contains the p-values of the Christoffersen (1998) tests of interval forecast conditional coverage (CC), interval forecast error independence (ID) and unconditional coverage (UC) for the quantile interval indicated in the first column. The row denoted "Joint" contains the result of a joint evaluation of interval forecasts which follows Section 4.2 in Christoffersen (1998) and aggregates the interval forecasts to six intervals spanning the quantiles (0.0 to 0.05], (0.05 to 0.25], (0.25 to 0.5], (0.5 to 0.75], (0.75 to 0.95], and (0.95 to 1.00]. For model evaluation purposes, we split our data into the estimation sample which contains 75% of the data (2,820 observations starting on Jan 2nd, 2004 and ending on April 7th, 2015), and the evaluation sample (the remaining 940 observations ending on December 31st, 2018). The p-values which indicate the rejection of the null hypotheses at the 0.05 level are printed in bold face. Those that indicate rejections at the 0.01 level are further marked with a $\star$. All model specifications contain a constant term and the regressor set according to the model label. $ES^{\mathbb{P}}$ is the non-parametric estimate of realized (physical) Expected Shortfall of intra-day S&P 500 returns before the close, on the day when each return calculation commences, $ES_t^{\mathbb{P}} = -E_t[R_{it}|R_{it} \leq F_{R_t}^{-1}(0.2)]$. Our Tail Risk Premium measure, $\Delta_{\mathbb{Q}}^{\mathbb{P}}ES$, is the difference between the risk-neutral and the physical ES. LTV is the Left Tail Variance of Bollerslev et al. (2015) obtained from `www.tailindex.com`. RV is the realized variance of intra-day returns on the S&P500 Index. The daily value of the VRP is defined as $VRP \equiv RV \times 365 - (VIX/100)^2$. "Both" indicates that the predictive model contains both the $ES^{\mathbb{P}}$ and $\Delta_{\mathbb{Q}}^{\mathbb{P}}ES$ terms.

	$ES^{\mathbb{P}}$			$\Delta_{\mathbb{Q}}^{\mathbb{P}}ES$			Both			$VRP + LTV$			$VRP + LTV + ES^{\mathbb{P}}$			$VRP + LTV + \Delta_{\mathbb{Q}}^{\mathbb{P}}ES$			$VRP + LTV$+Both		
Interval	CC	ID	UC	CC	ID	UC	CC	ID	UC	CC	ID	UC	CC	ID	UC	CC	ID	UC	CC	ID	UC
Joint	**0.02**	0.16	**0.00***	**0.00***	0.07	**0.00***	0.25	0.76	**0.01***	**0.00***	0.35	**0.00***	**0.02**	0.12	**0.01**	**0.00***	**0.02**	**0.01***	0.08	0.40	**0.01***
0.0 – 0.05	0.17	0.12	0.31	**0.00***	**0.01**	**0.01***	0.17	0.12	0.31	**0.00***	**0.05**	**0.00***	0.23	0.28	0.18	**0.01***	**0.00***	0.48	0.17	0.24	0.14
0.05 – 0.15	0.91	0.79	0.73	**0.00***	0.07	**0.00***	0.90	0.69	0.81	0.46	0.39	0.36	0.91	0.68	0.90	0.11	0.68	**0.04**	0.89	0.85	0.64
0.15 – 0.25	0.98	0.86	0.93	0.09	0.16	0.09	0.38	0.25	0.43	0.29	0.90	0.11	0.97	0.91	0.84	0.09	0.31	0.05	0.38	0.25	0.43
0.25 – 0.35	**0.01**	0.31	**0.00***	0.12	0.64	**0.05**	**0.01***	0.42	**0.00***	0.05	0.18	**0.05**	0.25	0.35	0.17	0.12	0.64	**0.05**	0.35	0.49	0.21
0.35 – 0.45	0.73	0.47	0.68	**0.00***	0.90	**0.00***	0.39	0.28	0.34	**0.00***	0.38	**0.00***	0.52	0.84	0.31	0.13	0.85	**0.04**	0.83	0.67	0.60
0.45 – 0.5	0.34	0.43	0.22	**0.02**	0.17	**0.01**	0.42	0.67	0.22	0.10	0.60	**0.04**	0.20	0.89	0.07	0.47	0.24	0.72	0.08	0.27	0.05
0.5 – 0.55	0.47	0.24	0.72	0.10	0.58	**0.04**	0.11	**0.05**	0.48	0.35	0.20	0.51	0.72	0.95	0.43	0.16	0.33	0.10	0.08	**0.02**	0.83
0.55 – 0.65	0.23	0.82	0.09	0.74	0.57	0.60	0.34	0.96	0.14	0.63	0.34	0.98	0.83	0.80	0.57	0.97	0.97	0.81	0.34	0.18	0.57
0.65 – 0.75	**0.03**	0.51	**0.01**	0.69	0.71	0.43	0.12	0.18	0.11	0.65	0.79	0.36	0.31	0.36	0.21	0.65	0.79	0.36	0.24	0.57	0.11
0.75 – 0.85	0.97	0.80	0.98	0.57	0.34	0.64	0.89	0.85	0.64	0.19	0.87	0.07	0.09	0.63	**0.03**	0.14	0.31	0.09	0.75	0.74	0.49
0.85 – 0.95	0.59	0.87	0.31	0.32	0.31	0.26	0.99	0.92	0.98	0.90	0.75	0.76	0.99	0.92	0.98	0.74	0.52	0.68	0.89	0.85	0.64
0.95 – 1.0	0.06	0.24	**0.04**	**0.00***	0.09	**0.00***	0.42	0.47	0.28	**0.01***	0.26	**0.00***	**0.00***	0.28	**0.00***	0.15	0.16	0.17	**0.01***	0.14	**0.01***

Table 5: Predictive regressions: S&P 500 returns with HF option data

The table reports regression coefficients and their standard deviations (in parentheses) for predictive regressions of the close-to-close S&P 500 returns at the 1-day horizon with $\Delta_{\mathbb{Q}}^{\mathbb{P}} ES$ calculated with the use of risk-neutral probabilities obtained from the joint risk-neutralization of high-frequency returns on the S&P 500 index and on options thereon, with option returns grouped by Δ. Δ01 corresponds to deep out of the money options while Δ05 corresponds to at the money options. The coefficients that are significant at the 0.05 confidence level are printed in bold and those significant at the 0.01 level are additionally highlighted with a $\star$. We report Andrews (1991) standard errors calculated with the use of the `sandwich 3.0.0` package for `R 4.0.3` (Zeileis et al., 2020). $ES^{\mathbb{P}}$ is the non-parametric estimate of realized (physical) Expected Shortfall of intra-day S&P 500 returns before the close, on the day when each return calculation commences, $ES_t^{\mathbb{P}} = -E_t[R_{it}|R_{it} \leq F_{R_t}^{-1}(0.2)]$. Our Tail Risk Premium measure, $\Delta_{\mathbb{Q}}^{\mathbb{P}} ES$, is the difference between the risk-neutral and the physical ES.

	CΔ01	CΔ02	CΔ03	CΔ04	CΔ05	PΔ01	PΔ02	PΔ03	PΔ04	PΔ05
$\Delta_{\mathbb{Q}}^{\mathbb{P}} ES$	**6.87***	3.88	**6.09***	2.50	2.23	4.22	3.37	3.09	1.98	2.67
	(2.46)	(2.22)	(1.90)	(1.62)	(1.68)	(2.93)	(2.71)	(2.43)	(2.78)	(2.73)
$ES^{\mathbb{P}}$	−0.08	0.02	−0.05	0.09	0.03	−0.01	−0.03	−0.01	0.02	0.04
	(0.78)	(0.77)	(0.79)	(0.81)	(0.80)	(0.82)	(0.84)	(0.81)	(0.86)	(0.84)
Constant	−0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
R^2 (adj)	0.0065	0.0027	0.0067	0.0016	0.0011	0.0026	0.0019	0.0017	0.00077	0.0017

Appendix

A Alternative CR Discrepancy

This Appendix reproduces our central results for the Hellinger case ($\gamma = -0.5$).

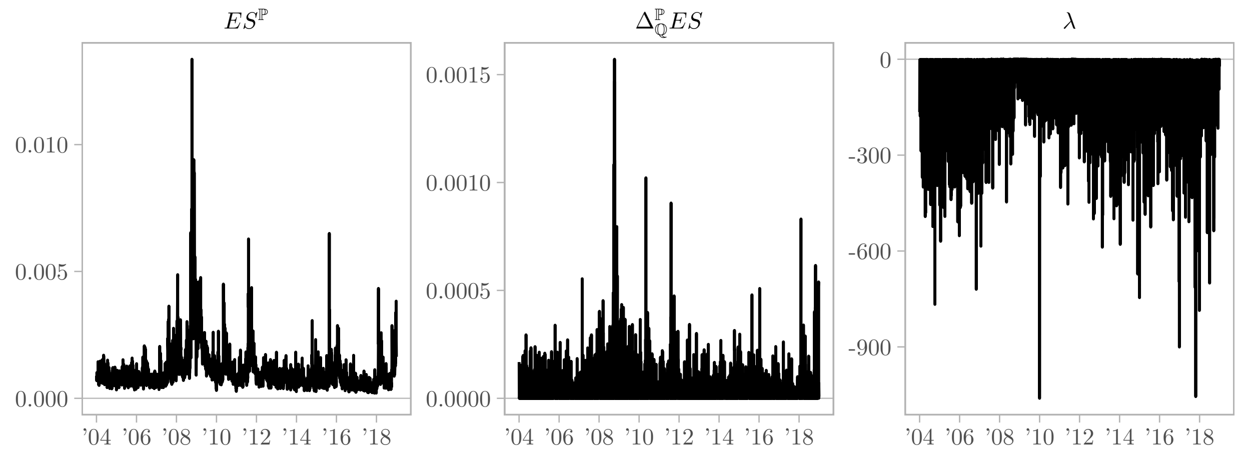


Figure A.1: Time series of tail measures implied by the intraday SDFs.

Table A.1: Robustness of Risk Premiums Predictions: $\gamma = -0.5$

The table reports regression coefficients and their standard deviations (in parentheses) for predictive regressions of the close-to-close S&P 500 excess returns and of the Variance Risk Premium at the 1-, 5- and 21-day horizons. The Variance Risk Premium is measured as the difference between: realized variance, measured from the intra-day S&P 500 returns, and the variance swap rate, measured with the use of the VIX index. The coefficients that are significant at the 0.05 confidence level are printed in bold and those significant at the 0.01 level are additionally highlighted with a $\star$. The 5- and 21-day horizon regressions are estimated on overlapping observations using the approach of Britten-Jones et al. (2011). We report Andrews (1991) standard errors calculated with the use of the `sandwich 3.0.0` package for `R 4.0.3` (Zeileis et al., 2020). $ES^{\mathbb{P}}$ is the non-parametric estimate of realized (physical) Expected Shortfall of intra-day S&P 500 returns before the close, on the day when each return calculation commences, $ES^{\mathbb{P}}_t = -E_t[R_{it}|R_{it} \leq F^{-1}_{R_t}(0.2)]$. Our Tail Risk Premium measure, $\Delta^{\mathbb{P}}_{\mathbb{Q}}ES$, is the difference between the risk-neutral and the physical ES. LTV is the Left Tail Variance of Bollerslev et al. (2015) obtained from `www.tailindex.com`. RV is the realized variance of intra-day returns on the S&P500 Index. IV is an estimate of integrated quadratic variation, the continuous component of realized variance, estimated as in Mancini and Gobbi (2012). JV is the jump component of realized variance, calculated as $\max\{RV - IV, 0\}$. RV, IV, and JV are calculated from return data sampled at the 5-minute frequency. The risk neutralization algorithm used the Hellinger estimator with $\gamma = -\frac{1}{2}$ in the Cressie Read Minimum Discrepancy problem of Section 2.3.

Panel A: Equity Risk Premium (Excess Return on the S&P 500 Index)

	1 days	5 days	21 days	1 days	5 days	21 days	1 days	5 days	21 days	1 days	5 days	21 days	1 days	5 days	21 days
$ES^{\mathbb{P}}$	0.17	0.51	−1.67	−0.19	−0.03	−2.20	−2.15	−2.71	−5.14	−2.12	−2.73	−5.07			
	(0.77)	(2.88)	(10.08)	(0.87)	(2.76)	(9.41)	(1.14)	(4.44)	(13.82)	(1.12)	(4.39)	(13.82)			
$\Delta^{\mathbb{P}}_{\mathbb{Q}}ES$				**11.01**	16.73	17.29	**12.57***	12.65	5.58	**12.42***	12.61	9.71	**10.42***	16.81	13.49
				(4.44)	(12.27)	(28.97)	(4.18)	(11.42)	(26.93)	(4.00)	(10.76)	(26.74)	(4.04)	(15.93)	(47.45)
LTV							−0.10	−0.24	−0.48	−0.10	−0.24	−0.49			
							(0.07)	(0.28)	(0.70)	(0.07)	(0.28)	(0.71)			
RV							15.68	28.62	50.65						
							(10.36)	(39.94)	(121.97)						
VRP							**−16.18**	−26.85	−82.53	**−16.35**	−27.35	−81.15	1.48	−2.03	−27.52
							(7.79)	(25.70)	(81.90)	(7.81)	(25.72)	(83.99)	(9.06)	(17.70)	(42.65)
JV										18.76	30.52	1.85			
										(14.39)	(44.89)	(150.72)			
IV										15.16	28.60	54.44			
										(10.75)	(40.69)	(121.75)			
Constant	0.00	0.00	0.01	−0.00	0.00	0.01	0.00	0.00	0.01	0.00	0.00	0.01	−0.00	0.00	0.00
	(0.00)	(0.00)	(0.01)	(0.00)	(0.00)	(0.01)	(0.00)	(0.00)	(0.01)	(0.00)	(0.00)	(0.01)	(0.00)	(0.00)	(0.00)
R^2 (adj)	−2.4e-06	−9.4e-05	−9e-05	0.0067	0.003	0.00091	0.02	0.0047	0.0057	0.019	0.0047	0.0055	0.0068	0.0024	0.00098

Panel B: Variance Risk Premium

	1 days	5 days	21 days	1 days	5 days	21 days	1 days	5 days	21 days	1 days	5 days	21 days
$ES^{\mathbb{P}}$	0.02	0.05	−0.16	−0.00	0.01	−0.23	**−0.02***	**−0.10***	**−0.57***	−0.03	−0.07	−0.22
	(0.01)	(0.07)	(0.24)	(0.01)	(0.05)	(0.17)	(0.01)	(0.04)	(0.16)	(0.02)	(0.06)	(0.23)
$\Delta^{\mathbb{P}}_{\mathbb{Q}}ES$				**0.54***	**1.19***	**2.06***	**0.57***	**1.35***	**2.60***	**0.31***	**0.76***	**1.94***
				(0.20)	(0.41)	(0.77)	(0.20)	(0.33)	(0.79)	(0.09)	(0.17)	(0.47)
VRP							**0.32***	**1.59***	**5.14***	**0.37***	**1.68***	**6.35***
							(0.07)	(0.18)	(0.58)	(0.10)	(0.43)	(1.21)
LTV										0.00	0.00	−0.01
										(0.00)	(0.00)	(0.01)
RV										0.02	−0.20	−1.87
										(0.11)	(0.60)	(2.05)
Constant	**−0.00***	**−0.00***	**−0.00***	**−0.00***	**−0.00***	**−0.00***	**−0.00**	**−0.00**	−0.00	−0.00	**−0.00***	**−0.00***
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
R^2 (adj)	0.092	0.087	0.087	0.22	0.21	0.21	0.29	0.32	0.25	0.3	0.32	0.3

B No Equity Risk Premium Beliefs

In this section, we present corresponding figures and tables for the tail risk premium series obtained with no restriction imposed to the equity premium of market returns, $\mathbb{E}(R - R_F)$, where R represents the market returns and R_F the risk-free rate. We measure the equity premium per day by calculating the above-mentioned expectation using intra-daily high-frequency S&P500 returns.

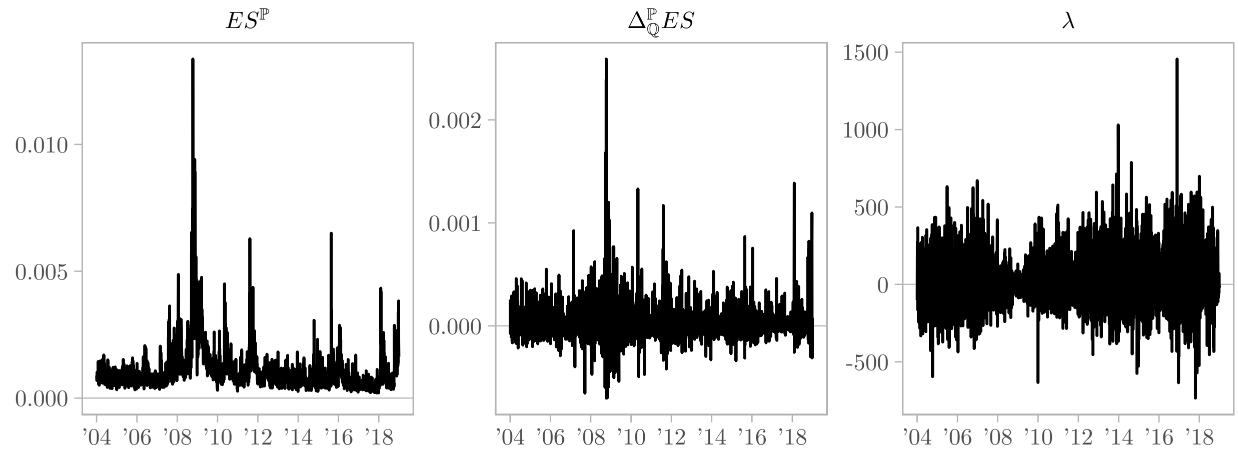


Figure B.1: Time series of tail measures implied by the intraday SDFs.

Table B.1: Robustness of Risk Premiums Predictions: no ERP Restriction

The table reports regression coefficients and their standard deviations (in parentheses) for predictive regressions of the close-to-close S&P 500 excess returns and of the Variance Risk Premium at the 1-, 5- and 21-day horizons. The Variance Risk Premium is measured as the difference between: realized variance, measured from the intra-day S&P 500 returns, and the variance swap rate, measured with the use of the VIX index. The coefficients that are significant at the 0.05 confidence level are printed in bold and those significant at the 0.01 level are additionally highlighted with a $\star$. The 5- and 21-day horizon regressions are estimated on overlapping observations using the approach of Britten-Jones et al. (2011). We report Andrews (1991) standard errors calculated with the use of the `sandwich 3.0.0` package for `R 4.0.3` (Zeileis et al., 2020). $ES^{\mathbb{P}}$ is the non-parametric estimate of realized (physical) Expected Shortfall of intra-day S&P 500 returns before the close, on the day when each return calculation commences, $ES_t^{\mathbb{P}} = -E_t[R_{it}|R_{it} \leq F_{R_t}^{-1}(0.2)]$. Our Tail Risk Premium measure, $\Delta_{\mathbb{Q}}^{\mathbb{P}}ES$, is the difference between the risk-neutral and the physical ES. LTV is the Left Tail Variance of Bollerslev et al. (2015) obtained from www.tailindex.com. RV is the realized variance of intra-day returns on the S&P500 Index. IV is an estimate of integrated quadratic variation, the continuous component of realized variance, estimated as in Mancini and Gobbi (2012). JV is the jump component of realized variance, calculated as $\max\{RV - IV, 0\}$. RV, IV, and JV are calculated from return data sampled at the 5-minute frequency. The risk neutralization algorithm of Section 2.3 uses $\gamma = -3$ as in the main text, but does not impose the constraint that the equity premium is non-negative.

Panel A: Equity Risk Premium (Excess Return on the S&P 500 Index)

	1 days	5 days	21 days	1 days	5 days	21 days	1 days	5 days	21 days	1 days	5 days	21 days	1 days	5 days	21 days
$ES^{\mathbb{P}}$	0.17	0.51	−1.67	0.13	0.43	−1.74	−1.68	−1.87	−4.46	−1.65	−1.89	−4.30			
	(0.77)	(2.88)	(10.08)	(0.84)	(2.91)	(10.04)	(1.21)	(4.70)	(14.86)	(1.19)	(4.59)	(14.63)			
$\Delta_{\mathbb{Q}}^{\mathbb{P}}ES$				**5.44**	10.42	12.06	**6.28***	8.97	6.00	**6.18***	8.93	7.60	**5.55**	10.48	10.75
				(2.39)	(6.77)	(15.54)	(2.22)	(5.72)	(14.73)	(2.08)	(5.35)	(14.42)	(2.22)	(7.66)	(18.47)
LTV							−0.11	−0.24	−0.49	−0.11	−0.25	−0.50			
							(0.07)	(0.28)	(0.70)	(0.08)	(0.27)	(0.70)			
RV							15.23	26.04	47.71						
							(10.61)	(40.47)	(124.23)						
VRP							**−16.05**	−25.46	−80.84	**−16.25**	−25.96	−79.65	2.22	−0.77	−26.36
							(7.72)	(26.07)	(82.70)	(7.79)	(25.80)	(83.40)	(9.01)	(18.11)	(45.18)
JV										20.21	28.31	−1.00			
										(13.77)	(44.58)	(147.36)			
IV										14.47	26.01	51.81			
										(11.24)	(40.81)	(121.74)			
Constant	0.00	0.00	0.01	−0.00	0.00	0.01	0.00	0.00	0.01	0.00	0.00	0.01	0.00	0.00	0.00
	(0.00)	(0.00)	(0.01)	(0.00)	(0.00)	(0.01)	(0.00)	(0.00)	(0.01)	(0.00)	(0.00)	(0.01)	(0.00)	(0.00)	(0.00)
R^2 (adj)	−2.4e-06	−9.4e-05	−9e-05	0.0058	0.0044	0.0018	0.019	0.0068	0.0077	0.018	0.0068	0.0075	0.0063	0.0047	0.0033

Panel B: Variance Risk Premium

	1 days	5 days	21 days	1 days	5 days	21 days	1 days	5 days	21 days	1 days	5 days	21 days
$ES^{\mathbb{P}}$	0.02	0.05	−0.16	0.01	0.04	−0.17	−0.01	−0.06	**−0.50***	−0.02	−0.04	−0.14
	(0.01)	(0.07)	(0.24)	(0.01)	(0.05)	(0.19)	(0.01)	(0.04)	(0.18)	(0.01)	(0.06)	(0.24)
$\Delta_{\mathbb{Q}}^{\mathbb{P}}ES$				**0.27**	**0.65***	**1.04**	**0.28***	**0.72***	**1.29***	**0.14***	**0.41***	**1.04***
				(0.11)	(0.22)	(0.41)	(0.10)	(0.18)	(0.42)	(0.05)	(0.09)	(0.25)
VRP							**0.31***	**1.58***	**5.11***	**0.37***	**1.70***	**6.41***
							(0.07)	(0.17)	(0.59)	(0.10)	(0.43)	(1.18)
LTV										0.00	0.00	−0.01
										(0.00)	(0.00)	(0.01)
RV										0.02	−0.25	−2.01
										(0.11)	(0.60)	(2.03)
Constant	**−0.00***	**−0.00***	**−0.00***	**−0.00***	**−0.00***	**−0.00***	**−0.00**	**−0.00**	−0.00	−0.00	**−0.00***	**−0.00***
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
R^2 (adj)	0.092	0.087	0.087	0.2	0.22	0.21	0.27	0.32	0.24	0.28	0.32	0.3

C Alternative threshold for ES calculation

In this section, we present the results for $ES^{\mathbb{P}}$ and $\Delta^{\mathbb{P}}_{\mathbb{Q}}ES$ calculated below the 10th percentile of intraday physical return realizations.

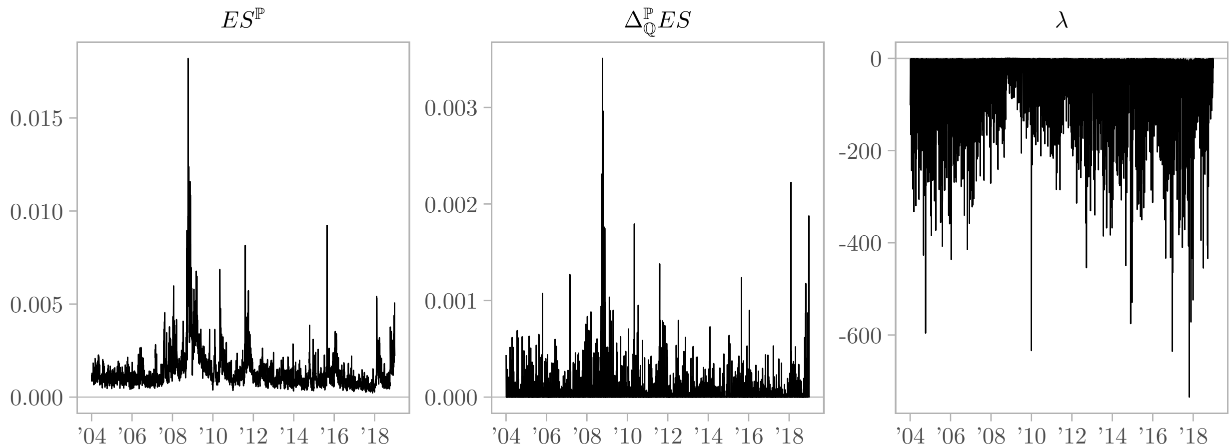


Figure C.1: Time series of tail measures implied by the intraday SDFs.

Table C.1: Robustness of Risk Premiums Predictions: ES in exceedance of 10th percentile

The table reports regression coefficients and their standard deviations (in parentheses) for predictive regressions of the close-to-close S&P 500 excess returns and of the Variance Risk Premium at the 1-, 5- and 21-day horizons. The Variance Risk Premium is measured as the difference between: realized variance, measured from the intra-day S&P 500 returns, and the variance swap rate, measured with the use of the VIX index. The coefficients that are significant at the 0.05 confidence level are printed in bold and those significant at the 0.01 level are additionally highlighted with a ⋆. The 5- and 21-day horizon regressions are estimated on overlapping observations using the approach of Britten-Jones et al. (2011). We report Andrews (1991) standard errors calculated with the use of the `sandwich 3.0.0` package for `R 4.0.3` (Zeileis et al., 2020). $ES^{\mathbb{P}}$ is the non-parametric estimate of realized (physical) Expected Shortfall of intra-day S&P 500 returns before the close, on the day when each return calculation commences, $ES^{\mathbb{P}}_t = -E_t[R_{it}|R_{it} \leq F^{-1}_{R_t}(0.1)]$. Our Tail Risk Premium measure, $\Delta^{\mathbb{P}}_{\mathbb{Q}}ES$, is the difference between the risk-neutral and the physical ES. LTV is the Left Tail Variance of Bollerslev et al. (2015) obtained from `www.tailindex.com`. RV is the realized variance of intra-day returns on the S&P500 Index. IV is an estimate of integrated quadratic variation, the continuous component of realized variance, estimated as in Mancini and Gobbi (2012). JV is the jump component of realized variance, calculated as $\max\{RV - IV, 0\}$. RV, IV, and JV are calculated from return data sampled at the 5-minute frequency.

Panel A: Equity Risk Premium (Excess Return on the S&P 500 Index)

	1 days	5 days	21 days	1 days	5 days	21 days	1 days	5 days	21 days	1 days	5 days	21 days	1 days	5 days	21 days
$ES^{\mathbb{P}}$	0.11	0.43	−1.27	−0.13	0.09	−1.55	**−1.76**	−1.75	−3.48	**−1.75**	−1.77	−3.14			
	(0.61)	(2.24)	(7.78)	(0.67)	(2.16)	(7.35)	(0.87)	(3.23)	(9.81)	(0.86)	(3.11)	(9.88)			
$\Delta^{\mathbb{P}}_{\mathbb{Q}}ES$				**5.80⋆**	8.36	7.37	**6.97⋆**	7.60	3.86	**6.85⋆**	7.56	6.34	**5.57⋆**	8.56	5.97
				(2.17)	(6.16)	(14.10)	(2.25)	(5.67)	(11.09)	(2.11)	(5.58)	(11.71)	(2.04)	(7.80)	(22.20)
LTV							−0.11	−0.24	−0.50	−0.11	−0.25	−0.50			
							(0.07)	(0.28)	(0.70)	(0.07)	(0.28)	(0.71)			
RV							16.60	26.74	47.89						
							(9.86)	(38.20)	(114.54)						
VRP							**−16.39**	−25.79	−80.71	**−16.63**	−26.27	−78.71	1.54	−1.86	−27.14
							(7.48)	(25.00)	(77.74)	(7.71)	(24.79)	(80.02)	(9.14)	(17.67)	(42.68)
JV										20.12	28.50	−1.39			
										(14.14)	(46.23)	(152.72)			
IV										16.14	26.77	49.95			
										(10.25)	(38.20)	(114.19)			
Constant	0.00	0.00	0.01	0.00	0.00	0.01	0.00	0.00	0.01	0.00	0.00	0.01	−0.00	0.00	0.00
	(0.00)	(0.00)	(0.01)	(0.00)	(0.00)	(0.01)	(0.00)	(0.00)	(0.01)	(0.00)	(0.00)	(0.01)	(0.00)	(0.00)	(0.00)
R^2 (adj)	−3.4e-05	−8.1e-05	−9e-05	0.0074	0.003	0.00053	0.022	0.0049	0.0054	0.021	0.0048	0.0052	0.0075	0.0028	0.0011

Panel B: Variance Risk Premium

	1 days	5 days	21 days	1 days	5 days	21 days	1 days	5 days	21 days	1 days	5 days	21 days
$ES^{\mathbb{P}}$	0.01	0.04	−0.11	0.00	0.01	−0.15	**−0.01⋆**	**−0.07**	**−0.43⋆**	**−0.02**	−0.06	−0.16
	(0.01)	(0.05)	(0.19)	(0.01)	(0.04)	(0.14)	(0.01)	(0.03)	(0.13)	(0.01)	(0.04)	(0.16)
$\Delta^{\mathbb{P}}_{\mathbb{Q}}ES$				**0.27⋆**	**0.62⋆**	**1.08⋆**	**0.29⋆**	**0.69⋆**	**1.32⋆**	**0.16⋆**	**0.37⋆**	**1.00⋆**
				(0.10)	(0.22)	(0.38)	(0.10)	(0.17)	(0.39)	(0.05)	(0.09)	(0.19)
VRP							**0.31⋆**	**1.58⋆**	**5.15⋆**	**0.37⋆**	**1.66⋆**	**6.34⋆**
							(0.07)	(0.18)	(0.58)	(0.10)	(0.42)	(1.13)
LTV										0.00	0.00	−0.01
										(0.00)	(0.00)	(0.01)
RV										0.03	−0.16	−1.86
										(0.10)	(0.60)	(1.93)
Constant	**−0.00⋆**	**−0.00⋆**	**−0.00⋆**	**−0.00⋆**	**−0.00⋆**	**−0.00⋆**	**−0.00**	**−0.00**	−0.00	−0.00	**−0.00⋆**	**−0.00⋆**
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
R^2 (adj)	0.094	0.088	0.086	0.22	0.22	0.22	0.29	0.32	0.24	0.29	0.31	0.3