Online Appendix to the Paper: Optimal Portfolio Strategies in the Presence of Regimes in Asset Returns

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This document presents additional material related to the above paper.

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1 Estimation of Parameters

We assume that a monthly period, as used in our paper, is short enough such that we can reasonably match Hamilton (1989) transition probabilities with our densities of transition probabilities as follows:[1]

$$\text{Prob}\{Y_{t+1} = j | Y_t = i\} = P_{ij,\Delta t=1} = P_{ij} = \frac{\lambda_{ij}}{-\lambda_{ii}}\left(1 - e^{\lambda_{ii}}\right), i \neq j, \quad (1)$$

in which we have conveniently chosen the time unit as the same as the period length (that is, one month). We will then have the following identities:

$$\lambda_{ii} = \ln P_{ii}, \text{ and} \quad (2a)$$

$$\lambda_{ij} = -\frac{P_{ij} \ln P_{ii}}{1 - P_{ii}}. \quad (2b)$$

We collect the (constant) discrete-time probabilities in a matrix we call **P**:

$$\mathbf{P} = \begin{bmatrix} P_{11} & P_{12} & \dots & P_{1m} \\ P_{21} & P_{22} & \dots & P_{2m} \\ \dots & \dots & \dots & \dots \\ P_{m1} & P_{m2} & \dots & P_{mm} \end{bmatrix}. \quad (3)$$

The investor uses the filter also explained in Hamilton (1989) to infer the current regime and take her optimal consumption and investment decisions. This optimal inference relies on iterating the pair of equations below:

$$\widehat{\mathbf{Y}}_{t|t} = \frac{\widehat{\mathbf{Y}}_{t|t-1} \circ \eta_t}{\mathbf{1}^{\mathbf{T}}\left(\widehat{\mathbf{Y}}_{t|t-1} \circ \eta_t\right)}, \text{ and} \quad (4a)$$

$$\widehat{\mathbf{Y}}_{t+1|t} = \mathbf{P}^{\mathbf{T}}\widehat{\mathbf{Y}}_{t|t}, \quad (4b)$$

in which $\mathbf{1}$ represents here an mx1 vector of ones and the symbol $\circ$ denotes the element-by-element multiplication. $\widehat{\mathbf{Y}}_{t|t}$ and $\widehat{\mathbf{Y}}_{t+1|t}$ are also m x 1 vectors containing the updated probabilities of having each regime running respectively at times t and $t+1$, given the available information at time t. Finally, η_t is another m x 1 vector whose elements are the joint prob-

[1] Note that this would be exact if the period considered were the infinitesimally short period dt.

ability densities of the risky assets returns and predictor at time t, conditioned by being on each of the states. The first equation above uses the current information (*i.e.*, the risky assets returns and predictor value) at time t to update the probabilities of each of the states in this last period t. The second equation then uses this update of the economy regime to optimally estimate the probabilities of being in each state in the next period (next month).

In order to put in practice the iteration above, we need a starting point. Obviously, this can be naturally guessed by the investor, based on her beliefs of the initial state of the economy. In this study, we will start from the long-run regime probabilities (often called ergodic or unconditional probabilities), which are given by:

$$\widehat{\mathbf{Y}}_{\mathbf{1}|\mathbf{0}} = \left(A^T A\right)^{-1} A^T e_{m+1}, \tag{5}$$

where e_{m+1} denotes the last column vector of the identity matrix of order $m+1$ and A is an $(m+1)$ x m matrix in which the first m rows are the rows of $I_m - P^T$ (I_m is the identity matrix of order m) and the last row has only 1's.[2]

To estimate the predictor value, we followed the Ornstein–Uhlenbeck formal solution:

$$x_t = x_0 e^{-\theta t} + \mu(1 - e^{-\theta t}) + \sigma \int_0^t e^{-\theta(t-s)} dW_s, \tag{6}$$

where, x_t and x_0 respectively represent p_{t+1} and p_t. The mean reversion speed parameter θ can be denoted as κ_p and the mean μ as the row vector of predictor means conditioned by regime $\bar{\mathbb{P}}$. Since we are rebalancing with the same frequency of the observations (monthly), we can define $t = 1$. Also, for the estimation, the expectation can be considered as zero. Then, rewriting equation (6), we can find $p_{t+1} = p_t e^{-\kappa_p} + \bar{\mathbb{P}}(1 - e^{-\kappa_p})$. And from there, we obtain the predictor log-return $p_{t+1} - p_t e^{-\kappa_p}$ and the predictor mean log-return $\bar{\mathbb{P}}(1 - e^{-\kappa_p})$.

To find vector η_t, we recall that the processes followed by the assets and the predictor, if on

[2]The demonstration of this formula can be found at Hamilton (1989).

a single-regime economy, admit the solutions below:[3]

$$\begin{bmatrix} \ln \frac{S_{1,t+1}}{S_{1,t}} \\ \ln \frac{S_{2,t+1}}{S_{2,t}} \\ \dots \\ \ln \frac{S_{n,t+1}}{S_{n,t}} \\ p_{t+1} - p_t e^{-\kappa} \end{bmatrix} = \begin{bmatrix} \mu_{1,i} - \frac{1}{2} \sum_{k=1}^{n} \sigma_{1k,i}^2 \\ \mu_{2,i} - \frac{1}{2} \sum_{k=1}^{n} \sigma_{2k,i}^2 \\ \dots \\ \mu_{n,i} - \frac{1}{2} \sum_{k=1}^{n} \sigma_{nk,i}^2 \\ \bar{\mathbb{P}}(1 - e^{-\kappa}) \end{bmatrix} + \begin{bmatrix} \sigma_{11,i} & 0 & \dots & 0 & 0 \\ \sigma_{21,i} & \sigma_{22,i} & \dots & 0 & 0 \\ \dots & \dots & \dots & 0 & 0 \\ \sigma_{n1,i} & \sigma_{n2,i} & \dots & \sigma_{nn,i} & 0 \\ \sigma_{p1} & \sigma_{p2} & \dots & \sigma_{pn} & \sigma_{pp} \end{bmatrix} \begin{bmatrix} \Delta Z_{1,t} \\ \Delta Z_{2,t} \\ \dots \\ \Delta Z_{n,t} \\ \Delta Z_{p,t} \end{bmatrix}, \quad (7)$$

such that the element of vector η_t occupying row i is:[4]

$$\eta_{t,i} = \frac{1}{(2\pi)^{\frac{n+1}{2}} |\sigma_i \sigma_i{}^T|^{\frac{1}{2}}} \exp\left[-\frac{1}{2} \left(\mathbf{LR_t} - \mathbf{MLR_i} \right)^T \left(\sigma_i \sigma_i^T \right)^{-1} \left(\mathbf{LR_t} - \mathbf{MLR_i} \right) \right], \quad (8)$$

where $\mathbf{LR_t}$ is an $(n+1)$ x 1 vector of the n risky assets observed log-returns at period t together with the predictor change in the same period (*i.e.*, the left-hand side of equation (7)). On its turn, $\mathbf{MLR_i}$ stores the mean of log-returns together with the mean predictor change conditioned on the regime:

$$\mathbf{LR_t} = \begin{bmatrix} \ln \frac{S_{1,t+1}}{S_{1,t}} \\ \ln \frac{S_{2,t+1}}{S_{2,t}} \\ \dots \\ \ln \frac{S_{n,t+1}}{S_{n,t}} \\ p_{t+1} - p_t e^{-\kappa} \end{bmatrix} \quad \text{and} \quad \mathbf{MLR_i} = \begin{bmatrix} \mu_{1,i} - \frac{1}{2} \sum_{k=1}^{n} \sigma_{1k,i}^2 \\ \mu_{2,i} - \frac{1}{2} \sum_{k=1}^{n} \sigma_{2k,i}^2 \\ \dots \\ \mu_{n,i} - \frac{1}{2} \sum_{k=1}^{n} \sigma_{nk,i}^2 \\ \bar{\mathbb{P}}(1 - e^{-\kappa}) \end{bmatrix}. \quad (9)$$

Notice that we need to estimate the mean reversion parameter κ_p beforehand, since it appears in the left-hand side of equation (9). Since κ_p is not regime-dependent, we followed Campani and Garcia (2019) and ran a first-order auto-regression of the discretized observations of p_t to compute κ_p. After κ_p has been computed, the values of the remaining parameters can be estimated using the maximum likelihood estimation under Hamilton (1989) methodology.

[3] We have assumed that any regime switch will take place at the end of period, as it is the standard in discrete-time regime switching models.

[4] Matrix σ_i is an $(n+1)$ x $(n+1)$ matrix built of $\sigma_{s,i}$ concatenated with σ_p in the last row and zeros in the first n rows of its last column, such as in equation (7). The expression $|\sigma_i \sigma_i{}^T|$ denotes the determinant of matrix $\sigma_i \sigma_i^T$.

2 The System of Equations to the Problem with No Consumption

Equation a (independent term):[5]

$$
\begin{aligned}
A_0' \;=\;& -\beta + (1-\gamma)\,r + \left(B_p + \frac{1}{2}A_p^2\right)\sigma_{\mathbf{p}}\sigma_{\mathbf{p}}^{\mathbf{T}} + \frac{1-\gamma}{2\gamma}A_p^2\overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{p}}^{\mathbf{T}} \\
&+ \sum_{i=1}^{m}\left(C_{pi} + A_pA_i\right)\sigma_{\mathbf{p}}\sigma_{\mathbf{i},\pi}^{\mathbf{T}} + \frac{1-\gamma}{\gamma}A_p\sum_{i=1}^{m}A_i\overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{i},\pi}^{\mathbf{T}} \\
&+ \sum_{i=1}^{m}B_i\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{i},\pi}^{\mathbf{T}} + \sum_{i<j}C_{ij}\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{j},\pi}^{\mathbf{T}} + \frac{1}{2}\sum_{i,j=1}^{m}A_iA_j\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{j},\pi}^{\mathbf{T}} \\
&+ \frac{1-\gamma}{2\gamma}\sum_{i,j=1}^{m}A_iA_j\overline{\sigma}_{\mathbf{i},\pi}\overline{\sigma}_{\mathbf{j},\pi}^{\mathbf{T}} + \frac{1-\gamma}{2\gamma}r^2\mathbf{1}^{\mathbf{T}}\left[\left(\mathbf{V}\pi_\infty\right)\left(\mathbf{V}\pi_\infty\right)^{\mathbf{T}}\right]^{-1}\mathbf{1} \\
&- \frac{1-\gamma}{\gamma}rA_p\overline{\sigma}_{\mathbf{p}}\left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}}\mathbf{1} - \frac{1-\gamma}{\gamma}r\sum_{i=1}^{m}A_i\overline{\sigma}_{\mathbf{i},\pi}\left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}}\mathbf{1}.
\end{aligned}
\tag{10a}
$$

Equation b (p_t term):

$$
\begin{aligned}
A_p' \;=\;& -A_p\kappa_p + 2A_pB_p\sigma_{\mathbf{p}}\sigma_{\mathbf{p}}^{\mathbf{T}} + 2\frac{1-\gamma}{\gamma}A_pB_p\overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{p}}^{\mathbf{T}} + \sum_{i=1}^{m}\left(A_pC_{pi} + 2A_iB_p\right)\sigma_{\mathbf{p}}\sigma_{\mathbf{i},\pi}^{\mathbf{T}} \\
&+ \frac{1-\gamma}{\gamma}\sum_{i=1}^{m}\left(A_pC_{pi} + 2A_iB_p\right)\overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{i},\pi}^{\mathbf{T}} + \frac{1}{2}\sum_{i,j=1}^{m}\left(A_iC_{pj} + A_jC_{pi}\right)\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{j},\pi}^{\mathbf{T}} \\
&+ \frac{1-\gamma}{2\gamma}\sum_{i,j=1}^{m}\left(A_iC_{pj} + A_jC_{pi}\right)\overline{\sigma}_{\mathbf{i},\pi}\overline{\sigma}_{\mathbf{j},\pi}^{\mathbf{T}} - \frac{1-\gamma}{\gamma}r\mathbb{B}^{\mathbf{T}}\left[\left(\mathbf{V}\pi_\infty\right)\left(\mathbf{V}\pi_\infty\right)^{\mathbf{T}}\right]^{-1}\mathbf{1} \\
&+ \frac{1-\gamma}{\gamma}A_p\overline{\sigma}_{\mathbf{p}}\left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}}\mathbb{B} - 2\frac{1-\gamma}{\gamma}rB_p\overline{\sigma}_{\mathbf{p}}\left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}}\mathbf{1} \\
&+ \frac{1-\gamma}{\gamma}\sum_{i=1}^{m}A_i\overline{\sigma}_{\mathbf{i},\pi}\left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}}\mathbb{B} - \frac{1-\gamma}{\gamma}r\sum_{i=1}^{m}C_{pi}\overline{\sigma}_{\mathbf{i},\pi}\left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}}\mathbf{1}.
\end{aligned}
\tag{10b}
$$

[5] In all the following equations, $\mathbb{A}_{:,i}$ represents the i^{th} column of matrix $\mathbb{A}$.

Equation c ($\pi_{i,t}$, $i \in R$, terms):

$$\begin{aligned} A_i' \;=\; & A_p\kappa_p\bar{p}_i + A_pC_{pi}\sigma_{\mathbf{p}}\sigma_{\mathbf{p}}^{\mathbf{T}} + \frac{1-\gamma}{\gamma}A_pC_{pi}\overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{p}}^{\mathbf{T}} + \left(2A_pB_i + A_iC_{pi}\right)\sigma_{\mathbf{p}}\sigma_{\mathbf{i},\pi}^{\mathbf{T}} \\ & + \sum_{j\neq i}\left(A_pC_{ji} + A_jC_{pi}\right)\sigma_{\mathbf{p}}\sigma_{\mathbf{j},\pi}^{\mathbf{T}} + \frac{1-\gamma}{\gamma}\left(2A_pB_i + A_iC_{pi}\right)\overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{i},\pi}^{\mathbf{T}} \\ & + \frac{1-\gamma}{\gamma}\sum_{j\neq i}\left(A_pC_{ji} + A_jC_{pi}\right)\overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{j},\pi}^{\mathbf{T}} + \sum_{j=1}^{m}\lambda_{ij}A_j + 2A_iB_i\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{i},\pi}^{\mathbf{T}} \\ & + \sum_{j\neq i}\left(A_iC_{ji} + 2A_jB_i\right)\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{j},\pi}^{\mathbf{T}} + \frac{1}{2}\sum_{j,k\neq i}\left(A_jC_{ki} + A_kC_{ji}\right)\sigma_{\mathbf{j},\pi}\sigma_{\mathbf{k},\pi}^{\mathbf{T}} \\ & + 2\frac{1-\gamma}{\gamma}A_iB_i\overline{\sigma}_{\mathbf{i},\pi}\overline{\sigma}_{\mathbf{i},\pi}^{\mathbf{T}} + \frac{1-\gamma}{\gamma}\sum_{j\neq i}\left(A_iC_{ji} + 2A_jB_i\right)\overline{\sigma}_{\mathbf{i},\pi}\overline{\sigma}_{\mathbf{j},\pi}^{\mathbf{T}} \\ & + \frac{1-\gamma}{2\gamma}\sum_{j,k\neq i}\left(A_jC_{ki} + A_kC_{ji}\right)\overline{\sigma}_{\mathbf{j},\pi}\overline{\sigma}_{\mathbf{k},\pi}^{\mathbf{T}} - \frac{1-\gamma}{\gamma}r\mathbf{1}^{\mathbf{T}}\left[\left(\mathbf{V}\pi_\infty\right)\left(\mathbf{V}\pi_\infty\right)^{\mathbf{T}}\right]^{-\mathbf{1}}\mathbb{A}_{:,i} \\ & + \frac{1-\gamma}{\gamma}\left[A_p\overline{\sigma}_{\mathbf{p}}\left(\mathbf{V}\pi_\infty\right)^{-1}\mathbb{A}_{:,i} - rC_{pi}\overline{\sigma}_{\mathbf{p}}\left(\mathbf{V}\pi_\infty\right)^{-1}\mathbf{1} + \sum_{j=1}^{m}A_j\overline{\sigma}_{\mathbf{j},\pi}\left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}}\mathbb{A}_{:,i}\right] \\ & - 2\frac{1-\gamma}{\gamma}rB_i\overline{\sigma}_{\mathbf{i},\pi}\left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}}\mathbf{1} - \frac{1-\gamma}{\gamma}r\sum_{j\neq i}C_{ji}\overline{\sigma}_{\mathbf{j},\pi}\left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}}\mathbf{1}. \end{aligned} \tag{10c}$$

Equation d (p_t^2 term):

$$\begin{aligned} B_p' \;=\; & -2B_p\kappa_p + 2B_p^2\sigma_{\mathbf{p}}\sigma_{\mathbf{p}}^{\mathbf{T}} + 2\sum_{i=1}^{m}B_pC_{pi}\sigma_{\mathbf{p}}\sigma_{\mathbf{i},\pi}^{\mathbf{T}} + 2\frac{1-\gamma}{\gamma}\sum_{i=1}^{m}B_pC_{pi}\overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{i},\pi}^{\mathbf{T}} \\ & + \frac{1}{2}\sum_{i,j=1}^{m}C_{pi}C_{pj}\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{j},\pi}^{\mathbf{T}} + \frac{1-\gamma}{2\gamma}\sum_{i,j=1}^{m}C_{pi}C_{pj}\overline{\sigma}_{\mathbf{i},\pi}\overline{\sigma}_{\mathbf{j},\pi}^{\mathbf{T}} \\ & + \frac{1-\gamma}{2\gamma}\mathbb{B}^{\mathbf{T}}\left[\left(\mathbf{V}\pi_\infty\right)\left(\mathbf{V}\pi_\infty\right)^{\mathbf{T}}\right]^{-\mathbf{1}}\mathbb{B} + 2\frac{1-\gamma}{\gamma}B_p^2\overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{p}}^{\mathbf{T}} \\ & + 2\frac{1-\gamma}{\gamma}B_p\overline{\sigma}_{\mathbf{p}}\left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}}\mathbb{B} + \frac{1-\gamma}{\gamma}\sum_{i=1}^{m}C_{pi}\overline{\sigma}_{\mathbf{i},\pi}\left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}}\mathbb{B}. \end{aligned} \tag{10d}$$

Equation e ($\pi^2_{i,t}$, $i \in R$, terms):

$$
\begin{aligned}
B'_i \;=\; & C_{pi}\kappa_p\bar{p}_i + \frac{1}{2}C^2_{pi}\sigma_{\mathbf{p}}\sigma^{\mathbf{T}}_{\mathbf{p}} + \frac{1-\gamma}{2\gamma}C^2_{pi}\overline{\sigma}_{\mathbf{p}}\overline{\sigma}^{\mathbf{T}}_{\mathbf{p}} + 2C_{pi}B_i\sigma_{\mathbf{p}}\sigma^{\mathbf{T}}_{\mathbf{i},\pi} \\
& + 2\frac{1-\gamma}{\gamma}C_{pi}B_i\overline{\sigma}_{\mathbf{p}}\overline{\sigma}^{\mathbf{T}}_{\mathbf{i},\pi} + \sum_{j\neq i}C_{pi}C_{ji}\sigma_{\mathbf{p}}\sigma^{\mathbf{T}}_{\mathbf{j},\pi} + \frac{1-\gamma}{\gamma}\sum_{j\neq i}C_{pi}C_{ji}\overline{\sigma}_{\mathbf{p}}\overline{\sigma}^{\mathbf{T}}_{\mathbf{j},\pi} \\
& + 2\lambda_{ii}B_i + \sum_{j\neq i}\lambda_{ij}C_{ji} + 2B^2_i\sigma_{\mathbf{i},\pi}\sigma^{\mathbf{T}}_{\mathbf{i},\pi} + 2\frac{1-\gamma}{\gamma}B^2_i\overline{\sigma}_{\mathbf{i},\pi}\overline{\sigma}^{\mathbf{T}}_{\mathbf{i},\pi} \\
& + 2B_i\sum_{j\neq i}C_{ji}\sigma_{\mathbf{i},\pi}\sigma^{\mathbf{T}}_{\mathbf{j},\pi} + 2\frac{1-\gamma}{\gamma}B_i\sum_{j\neq i}C_{ji}\overline{\sigma}_{\mathbf{i},\pi}\overline{\sigma}^{\mathbf{T}}_{\mathbf{j},\pi} + \frac{1}{2}\sum_{j,k\neq i}C_{ji}C_{ki}\sigma_{\mathbf{j},\pi}\sigma^{\mathbf{T}}_{\mathbf{k},\pi} \\
& + \frac{1-\gamma}{2\gamma}\sum_{j,k\neq i}C_{ji}C_{ki}\overline{\sigma}_{\mathbf{j},\pi}\overline{\sigma}^{\mathbf{T}}_{\mathbf{k},\pi} + \frac{1-\gamma}{2\gamma}\mathbb{A}^{\mathbf{T}}_{:,i}\left[(\mathbf{V}\pi_\infty)(\mathbf{V}\pi_\infty)^{\mathbf{T}}\right]^{-\mathbf{1}}\mathbb{A}_{:,i} \\
& + \frac{1-\gamma}{\gamma}C_{pi}\overline{\sigma}_{\mathbf{p}}(\mathbf{V}\pi_\infty)^{-\mathbf{1}}\mathbb{A}_{:,i} + 2\frac{1-\gamma}{\gamma}B_i\overline{\sigma}_{\mathbf{i},\pi}(\mathbf{V}\pi_\infty)^{-\mathbf{1}}\mathbb{A}_{:,i} \\
& + \frac{1-\gamma}{\gamma}\sum_{j\neq i}C_{ji}\overline{\sigma}_{\mathbf{j},\pi}(\mathbf{V}\pi_\infty)^{-\mathbf{1}}\mathbb{A}_{:,i}.
\end{aligned}
\tag{10e}
$$

Equation f ($p_t\pi_{i,t}$, $i \in R$, terms):

$$
\begin{aligned}
C'_{pi} \;=\; & 2B_p\kappa_p\bar{p}_i - C_{pi}\kappa_p + 2B_pC_{pi}\sigma_{\mathbf{p}}\sigma^{\mathbf{T}}_{\mathbf{p}} + 2\frac{1-\gamma}{\gamma}B_pC_{pi}\overline{\sigma}_{\mathbf{p}}\overline{\sigma}^{\mathbf{T}}_{\mathbf{p}} + 4B_pB_i\sigma_{\mathbf{p}}\sigma^{\mathbf{T}}_{\mathbf{i},\pi} + C^2_{pi}\sigma_{\mathbf{p}}\sigma^{\mathbf{T}}_{\mathbf{i},\pi} \\
& + \sum_{j\neq i}(2B_pC_{ji} + C_{pi}C_{pj})\sigma_{\mathbf{p}}\sigma^{\mathbf{T}}_{\mathbf{j},\pi} + 4\frac{1-\gamma}{\gamma}B_pB_i\overline{\sigma}_{\mathbf{p}}\overline{\sigma}^{\mathbf{T}}_{\mathbf{i},\pi} + \frac{1-\gamma}{\gamma}C^2_{pi}\overline{\sigma}_{\mathbf{p}}\overline{\sigma}^{\mathbf{T}}_{\mathbf{i},\pi} \\
& + \frac{1-\gamma}{\gamma}\sum_{j\neq i}(2B_pC_{ji} + C_{pi}C_{pj})\overline{\sigma}_{\mathbf{p}}\overline{\sigma}^{\mathbf{T}}_{\mathbf{j},\pi} + \sum_{j=1}^{m}\lambda_{ij}C_{pj} + 2B_iC_{pi}\sigma_{\mathbf{i},\pi}\sigma^{\mathbf{T}}_{\mathbf{i},\pi} \\
& + 2\frac{1-\gamma}{\gamma}B_iC_{pi}\overline{\sigma}_{\mathbf{i},\pi}\overline{\sigma}^{\mathbf{T}}_{\mathbf{i},\pi} + \sum_{j\neq i}(2B_iC_{pj} + C_{pi}C_{ji})\sigma_{\mathbf{i},\pi}\sigma^{\mathbf{T}}_{\mathbf{j},\pi} \\
& + \frac{1-\gamma}{\gamma}\sum_{j\neq i}(2B_iC_{pj} + C_{pi}C_{ji})\overline{\sigma}_{\mathbf{i},\pi}\overline{\sigma}^{\mathbf{T}}_{\mathbf{j},\pi} + \frac{1}{2}\sum_{j,k\neq i}(C_{pj}C_{ki} + C_{pk}C_{ji})\sigma_{\mathbf{j},\pi}\sigma^{\mathbf{T}}_{\mathbf{k},\pi} \\
& \frac{1-\gamma}{2\gamma}\sum_{j,k\neq i}(C_{pj}C_{ki} + C_{pk}C_{ji})\overline{\sigma}_{\mathbf{j},\pi}\overline{\sigma}^{\mathbf{T}}_{\mathbf{k},\pi} + \frac{1-\gamma}{\gamma}(\mathbb{A}_{:,i})^{\mathbf{T}}\left[(\mathbf{V}\pi_\infty)(\mathbf{V}\pi_\infty)^{\mathbf{T}}\right]^{-\mathbf{1}}\mathbb{B} \\
& + \frac{1-\gamma}{\gamma}\left[2B_p\overline{\sigma}_{\mathbf{p}}(\mathbf{V}\pi_\infty)^{-\mathbf{1}}\mathbb{A}_{:,i} + C_{pi}\overline{\sigma}_{\mathbf{p}}(\mathbf{V}\pi_\infty)^{-\mathbf{1}}\mathbb{B} + \sum_{j=1}^{m}C_{pj}\overline{\sigma}_{\mathbf{j},\pi}(\mathbf{V}\pi_\infty)^{-\mathbf{1}}\mathbb{A}_{:,i}\right] \\
& + 2\frac{1-\gamma}{\gamma}B_i\overline{\sigma}_{\mathbf{i},\pi}(\mathbf{V}\pi_\infty)^{-\mathbf{1}}\mathbb{B} + \frac{1-\gamma}{\gamma}\sum_{j\neq i}C_{ji}\overline{\sigma}_{\mathbf{j},\pi}(\mathbf{V}\pi_\infty)^{-\mathbf{1}}\mathbb{B}.
\end{aligned}
\tag{10f}
$$

Equation g ($\pi_{i,t}\pi_{j,t}$, $i < j \in R$, terms):

$$
\begin{aligned}
C'_{ij} \;=\;& C_{pi}\kappa_p\bar{p}_j + C_{pj}\kappa_p\bar{p}_i + C_{pi}C_{pj}\sigma_{\mathbf{p}}\sigma_{\mathbf{p}}^{\mathbf{T}} + \frac{1-\gamma}{\gamma}C_{pi}C_{pj}\sigma_{\mathbf{p}}\overline{\sigma}_{\mathbf{p}}^{\mathbf{T}} \\
&+ (2B_iC_{pj} + C_{pi}C_{ij})\,\sigma_{\mathbf{p}}\sigma_{\mathbf{i},\pi}^{\mathbf{T}} + (2B_jC_{pi} + C_{pj}C_{ij})\,\sigma_{\mathbf{p}}\sigma_{\mathbf{j},\pi}^{\mathbf{T}} \\
&+ \frac{1-\gamma}{\gamma}\left[(2B_iC_{pj} + C_{pi}C_{ij})\,\overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{i},\pi}^{\mathbf{T}} + (2B_jC_{pi} + C_{pj}C_{ij})\,\overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{j},\pi}^{\mathbf{T}}\right] \\
&+ \sum_{k\neq i,j}(C_{pi}C_{kj} + C_{pj}C_{ki})\,\sigma_{\mathbf{p}}\sigma_{\mathbf{k},\pi}^{\mathbf{T}} + \frac{1-\gamma}{\gamma}\sum_{k\neq i,j}(C_{pi}C_{kj} + C_{pj}C_{ki})\,\overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{k},\pi}^{\mathbf{T}} \\
&+ 2\lambda_{ij}B_j + 2\lambda_{ji}B_i + \lambda_{ii}C_{ij} + \lambda_{jj}C_{ij} + \sum_{k\neq i,j}(\lambda_{ik}C_{kj} + \lambda_{jk}C_{ki}) \\
&+ 2B_iC_{ij}\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{i},\pi}^{\mathbf{T}} + 2B_jC_{ij}\sigma_{\mathbf{j},\pi}\sigma_{\mathbf{j},\pi}^{\mathbf{T}} + 2\frac{1-\gamma}{\gamma}\left[B_iC_{ij}\overline{\sigma}_{\mathbf{i},\pi}\overline{\sigma}_{\mathbf{i},\pi}^{\mathbf{T}} + B_jC_{ij}\overline{\sigma}_{\mathbf{j},\pi}\overline{\sigma}_{\mathbf{j},\pi}^{\mathbf{T}}\right] \\
&+ \left(4B_iB_j + C_{ij}^2\right)\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{j},\pi}^{\mathbf{T}} + \frac{1-\gamma}{\gamma}\left(4B_iB_j + C_{ij}^2\right)\overline{\sigma}_{\mathbf{i},\pi}\overline{\sigma}_{\mathbf{j},\pi}^{\mathbf{T}} \\
&+ \sum_{k\neq i,j}(2B_iC_{jk} + C_{ij}C_{ki})\,\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{k},\pi}^{\mathbf{T}} + \sum_{k\neq i,j}(2B_jC_{ik} + C_{ji}C_{kj})\,\sigma_{\mathbf{j},\pi}\sigma_{\mathbf{k},\pi}^{\mathbf{T}} \\
&+ \frac{1-\gamma}{\gamma}\left[\sum_{k\neq i,j}(2B_iC_{jk} + C_{ij}C_{ki})\,\overline{\sigma}_{\mathbf{i},\pi}\overline{\sigma}_{\mathbf{k},\pi}^{\mathbf{T}} + \sum_{k\neq i,j}(2B_jC_{ik} + C_{ji}C_{kj})\,\overline{\sigma}_{\mathbf{j},\pi}\overline{\sigma}_{\mathbf{k},\pi}^{\mathbf{T}}\right] \\
&+ \sum_{k<l\neq i,j}(C_{ki}C_{lj} + C_{kj}C_{li})\,\sigma_{\mathbf{k},\pi}\sigma_{\mathbf{l},\pi}^{\mathbf{T}} + \sum_{k\neq i,j}C_{ik}C_{jk}\sigma_{\mathbf{k},\pi}\sigma_{\mathbf{k},\pi}^{\mathbf{T}} \\
&+ \frac{1-\gamma}{\gamma}\left[\sum_{k<l\neq i,j}(C_{ki}C_{lj} + C_{kj}C_{li})\,\overline{\sigma}_{\mathbf{k},\pi}\overline{\sigma}_{\mathbf{l},\pi}^{\mathbf{T}} + \sum_{k\neq i,j}C_{ik}C_{jk}\overline{\sigma}_{\mathbf{k},\pi}\overline{\sigma}_{\mathbf{k},\pi}^{\mathbf{T}}\right] \\
&+ \frac{1-\gamma}{\gamma}(\mathbb{A}_{:,i})^{\mathbf{T}}\left[(\mathbf{V}\pi_\infty)(\mathbf{V}\pi_\infty)^{\mathbf{T}}\right]^{-1}\mathbb{A}_{:,j} \\
&+ \frac{1-\gamma}{\gamma}C_{pi}\overline{\sigma}_{\mathbf{p}}(\mathbf{V}\pi_\infty)^{-1}\mathbb{A}_{:,j} + \frac{1-\gamma}{\gamma}C_{pj}\overline{\sigma}_{\mathbf{p}}(\mathbf{V}\pi_\infty)^{-1}\mathbb{A}_{:,i} \\
&+ 2\frac{1-\gamma}{\gamma}B_i\overline{\sigma}_{\mathbf{i},\pi}(\mathbf{V}\pi_\infty)^{-1}\mathbb{A}_{:,j} + \frac{1-\gamma}{\gamma}\sum_{k\neq i}C_{ik}\overline{\sigma}_{\mathbf{k},\pi}(\mathbf{V}\pi_\infty)^{-1}\mathbb{A}_{:,j} \\
&+ 2\frac{1-\gamma}{\gamma}B_j\overline{\sigma}_{\mathbf{j},\pi}(\mathbf{V}\pi_\infty)^{-1}\mathbb{A}_{:,i} + \frac{1-\gamma}{\gamma}\sum_{k\neq j}C_{jk}\overline{\sigma}_{\mathbf{k},\pi}(\mathbf{V}\pi_\infty)^{-1}\mathbb{A}_{:,i}.
\end{aligned}
\tag{10g}
$$

3 The System of Equations to the Problem when $\psi = 1$

Equation a (independent term):

$$
\begin{aligned}
A_0' \;=\;& -\beta A_0 + (1-\gamma)\,\beta\,(\ln\beta - 1) + (1-\gamma)\,r \\
&+ \left(B_p + \frac{1}{2}A_p^2\right)\sigma_{\mathbf{p}}\sigma_{\mathbf{p}}^{\mathbf{T}} + \frac{1-\gamma}{2\gamma}A_p^2\overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{p}}^{\mathbf{T}} \\
&+ \sum_{i=1}^{m}\left(C_{pi} + A_pA_i\right)\sigma_{\mathbf{p}}\sigma_{\mathbf{i},\pi}^{\mathbf{T}} + \frac{1-\gamma}{\gamma}A_p\sum_{i=1}^{m}A_i\overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{i},\pi}^{\mathbf{T}} \\
&+ \sum_{i=1}^{m}B_i\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{i},\pi}^{\mathbf{T}} + \sum_{i<j}C_{ij}\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{j},\pi}^{\mathbf{T}} + \frac{1}{2}\sum_{i,j=1}^{m}A_iA_j\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{j},\pi}^{\mathbf{T}} \\
&+ \frac{1-\gamma}{2\gamma}\sum_{i,j=1}^{m}A_iA_j\overline{\sigma}_{\mathbf{i},\pi}\overline{\sigma}_{\mathbf{j},\pi}^{\mathbf{T}} + \frac{1-\gamma}{2\gamma}r^2\mathbf{1}^{\mathbf{T}}\left[\left(\mathbf{V}\pi_\infty\right)\left(\mathbf{V}\pi_\infty\right)^{\mathbf{T}}\right]^{-1}\mathbf{1} \\
&- \frac{1-\gamma}{\gamma}rA_p\overline{\sigma}_{\mathbf{p}}\left(\mathbf{V}\pi_\infty\right)^{-1}\mathbf{1} - \frac{1-\gamma}{\gamma}r\sum_{i=1}^{m}A_i\overline{\sigma}_{\mathbf{i},\pi}\left(\mathbf{V}\pi_\infty\right)^{-1}\mathbf{1}.
\end{aligned}
\tag{11a}
$$

Equation b (p_t term):

$$
\begin{aligned}
A_p' \;=\;& -A_p\kappa_p - \beta A_p + 2A_pB_p\sigma_{\mathbf{p}}\sigma_{\mathbf{p}}^{\mathbf{T}} + 2\frac{1-\gamma}{\gamma}A_pB_p\overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{p}}^{\mathbf{T}} + \sum_{i=1}^{m}\left(A_pC_{pi} + 2A_iB_p\right)\sigma_{\mathbf{p}}\sigma_{\mathbf{i},\pi}^{\mathbf{T}} \\
&+ \frac{1-\gamma}{\gamma}\sum_{i=1}^{m}\left(A_pC_{pi} + 2A_iB_p\right)\overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{i},\pi}^{\mathbf{T}} + \frac{1}{2}\sum_{i,j=1}^{m}\left(A_iC_{pj} + A_jC_{pi}\right)\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{j},\pi}^{\mathbf{T}} \\
&+ \frac{1-\gamma}{2\gamma}\sum_{i,j=1}^{m}\left(A_iC_{pj} + A_jC_{pi}\right)\overline{\sigma}_{\mathbf{i},\pi}\overline{\sigma}_{\mathbf{j},\pi}^{\mathbf{T}} - \frac{1-\gamma}{\gamma}r\mathbb{B}^{\mathbf{T}}\left[\left(\mathbf{V}\pi_\infty\right)\left(\mathbf{V}\pi_\infty\right)^{\mathbf{T}}\right]^{-1}\mathbf{1} \\
&+ \frac{1-\gamma}{\gamma}A_p\overline{\sigma}_{\mathbf{p}}\left(\mathbf{V}\pi_\infty\right)^{-1}\mathbb{B} - 2\frac{1-\gamma}{\gamma}rB_p\overline{\sigma}_{\mathbf{p}}\left(\mathbf{V}\pi_\infty\right)^{-1}\mathbf{1} \\
&+ \frac{1-\gamma}{\gamma}\sum_{i=1}^{m}A_i\overline{\sigma}_{\mathbf{i},\pi}\left(\mathbf{V}\pi_\infty\right)^{-1}\mathbb{B} - \frac{1-\gamma}{\gamma}r\sum_{i=1}^{m}C_{pi}\overline{\sigma}_{\mathbf{i},\pi}\left(\mathbf{V}\pi_\infty\right)^{-1}\mathbf{1}.
\end{aligned}
\tag{11b}
$$

Equation c ($\pi_{i,t}$, $i \in R$, terms):

$$\begin{aligned}
A_i' &= -\beta A_i + A_p \kappa_p \bar{p}_i + A_p C_{pi} \sigma_{\mathbf{p}} \sigma_{\mathbf{p}}^{\mathbf{T}} + \frac{1-\gamma}{\gamma} A_p C_{pi} \overline{\sigma}_{\mathbf{p}} \overline{\sigma}_{\mathbf{p}}^{\mathbf{T}} + \left(2A_p B_i + A_i C_{pi}\right) \sigma_{\mathbf{p}} \sigma_{\mathbf{i},\pi}^{\mathbf{T}} \\
&+ \sum_{j \neq i} \left(A_p C_{ji} + A_j C_{pi}\right) \sigma_{\mathbf{p}} \sigma_{\mathbf{j},\pi}^{\mathbf{T}} + \frac{1-\gamma}{\gamma} \left(2A_p B_i + A_i C_{pi}\right) \overline{\sigma}_{\mathbf{p}} \overline{\sigma}_{\mathbf{i},\pi}^{\mathbf{T}} \\
&+ \frac{1-\gamma}{\gamma} \sum_{j \neq i} \left(A_p C_{ji} + A_j C_{pi}\right) \overline{\sigma}_{\mathbf{p}} \overline{\sigma}_{\mathbf{j},\pi}^{\mathbf{T}} + \sum_{j=1}^{m} \lambda_{ij} A_j + 2A_i B_i \sigma_{\mathbf{i},\pi} \sigma_{\mathbf{i},\pi}^{\mathbf{T}} \\
&+ \sum_{j \neq i} \left(A_i C_{ji} + 2A_j B_i\right) \sigma_{\mathbf{i},\pi} \sigma_{\mathbf{j},\pi}^{\mathbf{T}} + \frac{1}{2} \sum_{j,k \neq i} \left(A_j C_{ki} + A_k C_{ji}\right) \sigma_{\mathbf{j},\pi} \sigma_{\mathbf{k},\pi}^{\mathbf{T}} \\
&+ 2\frac{1-\gamma}{\gamma} A_i B_i \overline{\sigma}_{\mathbf{i},\pi} \overline{\sigma}_{\mathbf{i},\pi}^{\mathbf{T}} + \frac{1-\gamma}{\gamma} \sum_{j \neq i} \left(A_i C_{ji} + 2A_j B_i\right) \overline{\sigma}_{\mathbf{i},\pi} \overline{\sigma}_{\mathbf{j},\pi}^{\mathbf{T}} \\
&+ \frac{1-\gamma}{2\gamma} \sum_{j,k \neq i} \left(A_j C_{ki} + A_k C_{ji}\right) \overline{\sigma}_{\mathbf{j},\pi} \overline{\sigma}_{\mathbf{k},\pi}^{\mathbf{T}} - \frac{1-\gamma}{\gamma} r \mathbf{1}^{\mathbf{T}} \left[\left(\mathbf{V}\pi_\infty\right)\left(\mathbf{V}\pi_\infty\right)^{\mathbf{T}}\right]^{-\mathbf{1}} \mathbb{A}_{:,i} \\
&+ \frac{1-\gamma}{\gamma} \left[A_p \overline{\sigma}_{\mathbf{p}} \left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}} \mathbb{A}_{:,i} - r C_{pi} \overline{\sigma}_{\mathbf{p}} \left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}} \mathbf{1} + \sum_{j=1}^{m} A_j \overline{\sigma}_{\mathbf{j},\pi} \left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}} \mathbb{A}_{:,i}\right] \\
&- 2\frac{1-\gamma}{\gamma} r B_i \overline{\sigma}_{\mathbf{i},\pi} \left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}} \mathbf{1} - \frac{1-\gamma}{\gamma} r \sum_{j \neq i} C_{ji} \overline{\sigma}_{\mathbf{j},\pi} \left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}} \mathbf{1}.
\end{aligned} \tag{11c}$$

Equation d (p_t^2 term):

$$\begin{aligned}
B_p' &= -2B_p \kappa_p - \beta B_p + 2B_p^2 \sigma_{\mathbf{p}} \sigma_{\mathbf{p}}^{\mathbf{T}} + 2\sum_{i=1}^{m} B_p C_{pi} \sigma_{\mathbf{p}} \sigma_{\mathbf{i},\pi}^{\mathbf{T}} + 2\frac{1-\gamma}{\gamma} \sum_{i=1}^{m} B_p C_{pi} \overline{\sigma}_{\mathbf{p}} \overline{\sigma}_{\mathbf{i},\pi}^{\mathbf{T}} \\
&+ \frac{1}{2} \sum_{i,j=1}^{m} C_{pi} C_{pj} \sigma_{\mathbf{i},\pi} \sigma_{\mathbf{j},\pi}^{\mathbf{T}} + \frac{1-\gamma}{2\gamma} \sum_{i,j=1}^{m} C_{pi} C_{pj} \overline{\sigma}_{\mathbf{i},\pi} \overline{\sigma}_{\mathbf{j},\pi}^{\mathbf{T}} \\
&+ \frac{1-\gamma}{2\gamma} \mathbb{B}^{\mathbf{T}} \left[\left(\mathbf{V}\pi_\infty\right)\left(\mathbf{V}\pi_\infty\right)^{\mathbf{T}}\right]^{-\mathbf{1}} \mathbb{B} + 2\frac{1-\gamma}{\gamma} B_p^2 \overline{\sigma}_{\mathbf{p}} \overline{\sigma}_{\mathbf{p}}^{\mathbf{T}} \\
&+ 2\frac{1-\gamma}{\gamma} B_p \overline{\sigma}_{\mathbf{p}} \left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}} \mathbb{B} + \frac{1-\gamma}{\gamma} \sum_{i=1}^{m} C_{pi} \overline{\sigma}_{\mathbf{i},\pi} \left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}} \mathbb{B}.
\end{aligned} \tag{11d}$$

Equation e ($\pi^2_{i,t}$, $i \in R$, terms):

$$\begin{aligned}
B_i' &= -\beta B_i + C_{pi}\kappa_p\bar{p}_i + \frac{1}{2}C_{pi}^2\sigma_{\mathbf{p}}\sigma_{\mathbf{p}}^{\mathbf{T}} + \frac{1-\gamma}{2\gamma}C_{pi}^2\overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{p}}^{\mathbf{T}} + 2C_{pi}B_i\sigma_{\mathbf{p}}\sigma_{\mathbf{i},\pi}^{\mathbf{T}} \\
&+ 2\frac{1-\gamma}{\gamma}C_{pi}B_i\overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{i},\pi}^{\mathbf{T}} + \sum_{j\neq i}C_{pi}C_{ji}\sigma_{\mathbf{p}}\sigma_{\mathbf{j},\pi}^{\mathbf{T}} + \frac{1-\gamma}{\gamma}\sum_{j\neq i}C_{pi}C_{ji}\overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{j},\pi}^{\mathbf{T}} \\
&+ 2\lambda_{ii}B_i + \sum_{j\neq i}\lambda_{ij}C_{ji} + 2B_i^2\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{i},\pi}^{\mathbf{T}} + 2\frac{1-\gamma}{\gamma}B_i^2\overline{\sigma}_{\mathbf{i},\pi}\overline{\sigma}_{\mathbf{i},\pi}^{\mathbf{T}} \\
&+ 2B_i\sum_{j\neq i}C_{ji}\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{j},\pi}^{\mathbf{T}} + 2\frac{1-\gamma}{\gamma}B_i\sum_{j\neq i}C_{ji}\overline{\sigma}_{\mathbf{i},\pi}\overline{\sigma}_{\mathbf{j},\pi}^{\mathbf{T}} + \frac{1}{2}\sum_{j,k\neq i}C_{ji}C_{ki}\sigma_{\mathbf{j},\pi}\sigma_{\mathbf{k},\pi}^{\mathbf{T}} \\
&+ \frac{1-\gamma}{2\gamma}\sum_{j,k\neq i}C_{ji}C_{ki}\overline{\sigma}_{\mathbf{j},\pi}\overline{\sigma}_{\mathbf{k},\pi}^{\mathbf{T}} + \frac{1-\gamma}{2\gamma}\mathbb{A}_{:,i}^{\mathbf{T}}\left[(\mathbf{V}\pi_\infty)(\mathbf{V}\pi_\infty)^{\mathbf{T}}\right]^{-\mathbf{1}}\mathbb{A}_{:,i} \\
&+ \frac{1-\gamma}{\gamma}C_{pi}\overline{\sigma}_{\mathbf{p}}(\mathbf{V}\pi_\infty)^{-\mathbf{1}}\mathbb{A}_{:,i} + 2\frac{1-\gamma}{\gamma}B_i\overline{\sigma}_{\mathbf{i},\pi}(\mathbf{V}\pi_\infty)^{-\mathbf{1}}\mathbb{A}_{:,i} \\
&+ \frac{1-\gamma}{\gamma}\sum_{j\neq i}C_{ji}\overline{\sigma}_{\mathbf{j},\pi}(\mathbf{V}\pi_\infty)^{-\mathbf{1}}\mathbb{A}_{:,i}.
\end{aligned} \tag{11e}$$

Equation f ($p_t\pi_{i,t}$, $i \in R$, terms):

$$\begin{aligned}
C_{pi}' &= -\beta C_{pi} + 2B_p\kappa_p\bar{p}_i - C_{pi}\kappa_p + 2B_pC_{pi}\sigma_{\mathbf{p}}\sigma_{\mathbf{p}}^{\mathbf{T}} + 2\frac{1-\gamma}{\gamma}B_pC_{pi}\overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{p}}^{\mathbf{T}} + 4B_pB_i\sigma_{\mathbf{p}}\sigma_{\mathbf{i},\pi}^{\mathbf{T}} + C_{pi}^2\sigma_{\mathbf{p}}\sigma_{\mathbf{i},\pi}^{\mathbf{T}} \\
&+ \sum_{j\neq i}(2B_pC_{ji} + C_{pi}C_{pj})\sigma_{\mathbf{p}}\sigma_{\mathbf{j},\pi}^{\mathbf{T}} + 4\frac{1-\gamma}{\gamma}B_pB_i\overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{i},\pi}^{\mathbf{T}} + \frac{1-\gamma}{\gamma}C_{pi}^2\overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{i},\pi}^{\mathbf{T}} \\
&+ \frac{1-\gamma}{\gamma}\sum_{j\neq i}(2B_pC_{ji} + C_{pi}C_{pj})\overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{j},\pi}^{\mathbf{T}} + \sum_{j=1}^{m}\lambda_{ij}C_{pj} + 2B_iC_{pi}\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{i},\pi}^{\mathbf{T}} \\
&+ 2\frac{1-\gamma}{\gamma}B_iC_{pi}\overline{\sigma}_{\mathbf{i},\pi}\overline{\sigma}_{\mathbf{i},\pi}^{\mathbf{T}} + \sum_{j\neq i}(2B_iC_{pj} + C_{pi}C_{ji})\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{j},\pi}^{\mathbf{T}} \\
&+ \frac{1-\gamma}{\gamma}\sum_{j\neq i}(2B_iC_{pj} + C_{pi}C_{ji})\overline{\sigma}_{\mathbf{i},\pi}\overline{\sigma}_{\mathbf{j},\pi}^{\mathbf{T}} + \frac{1}{2}\sum_{j,k\neq i}(C_{pj}C_{ki} + C_{pk}C_{ji})\sigma_{\mathbf{j},\pi}\sigma_{\mathbf{k},\pi}^{\mathbf{T}} \\
&+ \frac{1-\gamma}{2\gamma}\sum_{j,k\neq i}(C_{pj}C_{ki} + C_{pk}C_{ji})\overline{\sigma}_{\mathbf{j},\pi}\overline{\sigma}_{\mathbf{k},\pi}^{\mathbf{T}} + \frac{1-\gamma}{\gamma}(\mathbb{A}_{:,i})^{\mathbf{T}}\left[(\mathbf{V}\pi_\infty)(\mathbf{V}\pi_\infty)^{\mathbf{T}}\right]^{-\mathbf{1}}\mathbb{B} \\
&+ \frac{1-\gamma}{\gamma}\left[2B_p\overline{\sigma}_{\mathbf{p}}(\mathbf{V}\pi_\infty)^{-\mathbf{1}}\mathbb{A}_{:,i} + C_{pi}\overline{\sigma}_{\mathbf{p}}(\mathbf{V}\pi_\infty)^{-\mathbf{1}}\mathbb{B} + \sum_{j=1}^{m}C_{pj}\overline{\sigma}_{\mathbf{j},\pi}(\mathbf{V}\pi_\infty)^{-\mathbf{1}}\mathbb{A}_{:,i}\right] \\
&+ 2\frac{1-\gamma}{\gamma}B_i\overline{\sigma}_{\mathbf{i},\pi}(\mathbf{V}\pi_\infty)^{-\mathbf{1}}\mathbb{B} + \frac{1-\gamma}{\gamma}\sum_{j\neq i}C_{ji}\overline{\sigma}_{\mathbf{j},\pi}(\mathbf{V}\pi_\infty)^{-\mathbf{1}}\mathbb{B}.
\end{aligned} \tag{11f}$$

Equation g ($\pi_{i,t}\pi_{j,t}$, $i < j \in R$, terms):

$$\begin{aligned}
C'_{ij} \; = \; & -\beta C_{ij} + C_{pi}\kappa_p \bar{p}_j + C_{pj}\kappa_p \bar{p}_i + C_{pi}C_{pj}\sigma_{\mathbf{p}}\sigma_{\mathbf{p}}^{\mathbf{T}} + \frac{1-\gamma}{\gamma} C_{pi}C_{pj}\sigma_{\mathbf{p}}\overline{\sigma}_{\mathbf{p}}^{\mathbf{T}} \\
& + \left(2B_i C_{pj} + C_{pi}C_{ij}\right)\sigma_{\mathbf{p}}\sigma_{\mathbf{i},\pi}^{\mathbf{T}} + \left(2B_j C_{pi} + C_{pj}C_{ij}\right)\sigma_{\mathbf{p}}\sigma_{\mathbf{j},\pi}^{\mathbf{T}} \\
& + \frac{1-\gamma}{\gamma}\left[\left(2B_i C_{pj} + C_{pi}C_{ij}\right)\overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{i},\pi}^{\mathbf{T}} + \left(2B_j C_{pi} + C_{pj}C_{ij}\right)\overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{j},\pi}^{\mathbf{T}}\right] \\
& + \sum_{k\neq i,j}\left(C_{pi}C_{kj} + C_{pj}C_{ki}\right)\sigma_{\mathbf{p}}\sigma_{\mathbf{k},\pi}^{\mathbf{T}} + \frac{1-\gamma}{\gamma}\sum_{k\neq i,j}\left(C_{pi}C_{kj} + C_{pj}C_{ki}\right)\overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{k},\pi}^{\mathbf{T}} \\
& + 2\lambda_{ij}B_j + 2\lambda_{ji}B_i + \lambda_{ii}C_{ij} + \lambda_{jj}C_{ij} + \sum_{k\neq i,j}\left(\lambda_{ik}C_{kj} + \lambda_{jk}C_{ki}\right) \\
& + 2B_i C_{ij}\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{i},\pi}^{\mathbf{T}} + 2B_j C_{ij}\sigma_{\mathbf{j},\pi}\sigma_{\mathbf{j},\pi}^{\mathbf{T}} + 2\frac{1-\gamma}{\gamma}\left[B_i C_{ij}\overline{\sigma}_{\mathbf{i},\pi}\overline{\sigma}_{\mathbf{i},\pi}^{\mathbf{T}} + B_j C_{ij}\overline{\sigma}_{\mathbf{j},\pi}\overline{\sigma}_{\mathbf{j},\pi}^{\mathbf{T}}\right] \\
& + \left(4B_iB_j + C_{ij}^2\right)\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{j},\pi}^{\mathbf{T}} + \frac{1-\gamma}{\gamma}\left(4B_iB_j + C_{ij}^2\right)\overline{\sigma}_{\mathbf{i},\pi}\overline{\sigma}_{\mathbf{j},\pi}^{\mathbf{T}} \\
& + \sum_{k\neq i,j}\left(2B_iC_{jk} + C_{ij}C_{ki}\right)\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{k},\pi}^{\mathbf{T}} + \sum_{k\neq i,j}\left(2B_jC_{ik} + C_{ji}C_{kj}\right)\sigma_{\mathbf{j},\pi}\sigma_{\mathbf{k},\pi}^{\mathbf{T}} \\
& + \frac{1-\gamma}{\gamma}\left[\sum_{k\neq i,j}\left(2B_iC_{jk} + C_{ij}C_{ki}\right)\overline{\sigma}_{\mathbf{i},\pi}\overline{\sigma}_{\mathbf{k},\pi}^{\mathbf{T}} + \sum_{k\neq i,j}\left(2B_jC_{ik} + C_{ji}C_{kj}\right)\overline{\sigma}_{\mathbf{j},\pi}\overline{\sigma}_{\mathbf{k},\pi}^{\mathbf{T}}\right] \\
& + \sum_{k<l\neq i,j}\left(C_{ki}C_{lj} + C_{kj}C_{li}\right)\sigma_{\mathbf{k},\pi}\sigma_{\mathbf{l},\pi}^{\mathbf{T}} + \sum_{k\neq i,j}C_{ik}C_{jk}\sigma_{\mathbf{k},\pi}\sigma_{\mathbf{k},\pi}^{\mathbf{T}} \\
& + \frac{1-\gamma}{\gamma}\left[\sum_{k<l\neq i,j}\left(C_{ki}C_{lj} + C_{kj}C_{li}\right)\overline{\sigma}_{\mathbf{k},\pi}\overline{\sigma}_{\mathbf{l},\pi}^{\mathbf{T}} + \sum_{k\neq i,j}C_{ik}C_{jk}\overline{\sigma}_{\mathbf{k},\pi}\overline{\sigma}_{\mathbf{k},\pi}^{\mathbf{T}}\right] \\
& + \frac{1-\gamma}{\gamma}\left(\mathbb{A}_{:,i}\right)^{\mathbf{T}}\left[\left(\mathbf{V}\pi_\infty\right)\left(\mathbf{V}\pi_\infty\right)^{\mathbf{T}}\right]^{-\mathbf{1}}\mathbb{A}_{:,j} \\
& + \frac{1-\gamma}{\gamma}C_{pi}\overline{\sigma}_{\mathbf{p}}\left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}}\mathbb{A}_{:,j} + \frac{1-\gamma}{\gamma}C_{pj}\overline{\sigma}_{\mathbf{p}}\left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}}\mathbb{A}_{:,i} \\
& + 2\frac{1-\gamma}{\gamma}B_i\overline{\sigma}_{\mathbf{i},\pi}\left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}}\mathbb{A}_{:,j} + \frac{1-\gamma}{\gamma}\sum_{k\neq i}C_{ik}\overline{\sigma}_{\mathbf{k},\pi}\left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}}\mathbb{A}_{:,j} \\
& + 2\frac{1-\gamma}{\gamma}B_j\overline{\sigma}_{\mathbf{j},\pi}\left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}}\mathbb{A}_{:,i} + \frac{1-\gamma}{\gamma}\sum_{k\neq j}C_{jk}\overline{\sigma}_{\mathbf{k},\pi}\left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}}\mathbb{A}_{:,i}.
\end{aligned} \tag{11g}$$

4 The Solution when $\psi \neq 1$

Under the general problem, the equation below is not linear due to its first term and therefore it is not analytically solvable.

$$
\begin{aligned}
0 \;=\; & \frac{1-\gamma}{\psi-1}\left(\beta^{\psi} e^{\frac{1-\psi}{1-\gamma} g}-\beta\psi\right)-A_0'-A_p' p_t-\sum_{i=1}^{m} A_i'\pi_{i,t}-B_p' p_t^2-\sum_{i=1}^{m} B_i'\pi_{i,t}^2-\sum_{i=1}^{m} C_{pi}' p_t\pi_{i,t} \\
& -\sum_{i<j} C_{ij}'\pi_{i,t}\pi_{j,t}+(1-\gamma)\, r \;+\left(A_p+2B_p p_t+\sum_{i=1}^{m} C_{pi}\pi_{i,t}\right) * \kappa_p(\bar{\mathbb{P}}\pi_t-p_t) \\
& +\frac{1}{2}\left[2B_p+\left(A_p+2B_p p_t+\sum_{i=1}^{m} C_{pi}\pi_{i,t}\right)^2\right]\sigma_p\sigma_p^{\mathbf{T}} \\
& +\sum_{i=1}^{m}\left[C_{pi}+\left(A_p+2B_p p_t+\sum_{k=1}^{m} C_{pk}\pi_{k,t}\right)\left(A_i+2B_i\pi_{i,t}+C_{pi}p_t+\sum_{k\neq i} C_{ik}\pi_{k,t}\right)\right]\sigma_p\sigma_{i,\pi}^{\mathbf{T}} \\
& +\frac{1-\gamma}{\gamma}\sum_{i=1}^{m}\left(A_p+2B_p p_t+\sum_{k=1}^{m} C_{pk}\pi_{k,t}\right)\left(A_i+2B_i\pi_{i,t}+C_{pi}p_t+\sum_{k\neq i} C_{ik}\pi_{k,t}\right)\overline{\sigma}_p\overline{\sigma}_{i,\pi}^{\mathbf{T}} \\
& +\sum_{i,j=1}^{m}\lambda_{ji}\pi_{j,t}\left(A_i+2B_i\pi_{i,t}+C_{pi}p_t+\sum_{k\neq i} C_{ik}\pi_{k,t}\right)+\sum_{i=1}^{m} B_i\sigma_{i,\pi}\sigma_{i,\pi}^{\mathbf{T}}+\sum_{i<j} C_{ij}\sigma_{i,\pi}\sigma_{j,\pi}^{\mathbf{T}} \\
& +\frac{1}{2}\sum_{i,j=1}^{m}\left(A_i+2B_i\pi_{i,t}+C_{pi}p_t+\sum_{k\neq i} C_{ik}\pi_{k,t}\right)\left(A_j+2B_j\pi_{j,t}+C_{pj}p_t+\sum_{k\neq j} C_{jk}\pi_{k,t}\right)\sigma_{i,\pi}\sigma_{j,\pi}^{\mathbf{T}} \\
& +\frac{1}{2}\frac{1-\gamma}{\gamma}\sum_{i,j=1}^{m}\left(A_i+2B_i\pi_{i,t}+C_{pi}p_t+\sum_{k\neq i} C_{ik}\pi_{k,t}\right)\left(A_j+2B_j\pi_{j,t}+C_{pj}p_t+\sum_{k\neq j} C_{jk}\pi_{k,t}\right)\overline{\sigma}_{i,\pi}\overline{\sigma}_{j,\pi}^{\mathbf{T}} \\
& +\frac{1}{2}\frac{1-\gamma}{\gamma}\left(\mathbb{A}\pi_t+\mathbb{B}p_t-r\mathbf{1}\right)^{\mathbf{T}}\left[\left(\mathbf{V}\pi_\infty\right)\left(\mathbf{V}\pi_\infty\right)^{\mathbf{T}}\right]^{-1}\left(\mathbb{A}\pi_t+\mathbb{B}p_t-r\mathbf{1}\right) \\
& +\frac{1}{2}\frac{1-\gamma}{\gamma}\left(A_p+2B_p p_t+\sum_{i=1}^{m} C_{pi}\pi_{i,t}\right)^2\overline{\sigma}_p\overline{\sigma}_p^{\mathbf{T}} \\
& +\frac{1-\gamma}{\gamma}\left(A_p+2B_p p_t+\sum_{i=1}^{m} C_{pi}\pi_{i,t}\right)\overline{\sigma}_p\left(\mathbf{V}\pi_\infty\right)^{-1}\left(\mathbb{A}\pi_t+\mathbb{B}p_t-r\mathbf{1}\right) \\
& +\frac{1-\gamma}{\gamma}\sum_{i=1}^{m}\left(A_i+2B_i\pi_{i,t}+C_{pi}p_t+\sum_{k\neq i} C_{ik}\pi_{k,t}\right)\overline{\sigma}_{i,\pi}\left(\mathbf{V}\pi_\infty\right)^{-1}\left(\mathbb{A}\pi_t+\mathbb{B}p_t-r\mathbf{1}\right).
\end{aligned}
\tag{12}
$$

However, this term turns out to be (proportional to) the consumption to wealth ratio, what can be shown by substituting the value function in the first-order condition for consumption:

$$
\frac{C_t}{W_t}=\beta^{\psi} e^{\frac{1-\psi}{1-\gamma} g(p_t,\pi_t,\tau)}\,. \tag{13}
$$

We will apply exactly the same methodology applied in Campani and Garcia (2019), based on log-linear techniques. The discussion there remains the same here: the point around which the approximation will be made can be chosen by the investor according to current conditions, beliefs and investment horizon. The state variable space is given here by the predictor and π_t. In this study, to be consistent with the volatility approximation made, we will approximate around the (log of) consumption to wealth ratio value under the ergodic probabilities. Regarding the predictor, we choose its historical unconditional mean (although any value could be chosen), denoted by $\overline{p}$:

$$\begin{aligned}
\beta^{\psi} e^{\frac{1-\psi}{1-\gamma} g(p_t,\pi_t,\tau)} &= \exp\left(\ln \frac{C_t}{W_t}\right) \\
&\approx \exp\left(\ln \frac{C_t}{W_t}\right)_{p_t=\overline{p},\pi_t=\pi_\infty} + \exp\left(\ln \frac{C_t}{W_t}\right)_{p_t=\overline{p},\pi_t=\pi_\infty} \left[\ln \frac{C_t}{W_t} - \left(\ln \frac{C_t}{W_t}\right)_{p_t=\overline{p},\pi_t=\pi_\infty}\right] \\
&= \beta^{\psi} e^{\frac{1-\psi}{1-\gamma} g(\overline{p},\pi_\infty,\tau)} * \left\{1 + \left[\ln \frac{C_t}{W_t} - \left(\ln \frac{C_t}{W_t}\right)_{p_t=\overline{p},\pi_t=\pi_\infty}\right]\right\}.
\end{aligned} \tag{14}$$

The approximation (14) renders equation (12) solvable, with a solution given by:

$$\begin{aligned}
g\left(p_t, \pi_t, \tau\right) \; &= A_0\left(\tau\right) + A_p\left(\tau\right) p_t + \sum_{i=1}^{m} A_i\left(\tau\right) \pi_{i,t} + B_p\left(\tau\right) p_t^2 \\
&+ \sum_{i=1}^{m} B_i\left(\tau\right) \pi_{i,t}^2 + \sum_{i=1}^{m} C_{pi}\left(\tau\right) p_t \pi_{i,t} + \sum_{i<j} C_{ij}\left(\tau\right) \pi_{i,t}\pi_{j,t}.
\end{aligned} \tag{15}$$

The partial differential equation to be solved is then the previous one with the log-linearization:

$$
\begin{aligned}
0 =& \beta^{\psi} e^{\frac{1-\psi}{1-\gamma}\left(A_0(\tau)+A_p(\tau)\overline{p}+\sum_{i=1}^{m} A_i(\tau)\pi_{i,\infty}+B_p(\tau)\overline{p}^2+\sum_{i=1}^{m} B_i(\tau)\pi_{i,\infty}^2+\sum_{i=1}^{m} C_{pi}(\tau)\overline{p}\pi_{i,\infty}+\sum_{i<j} C_{ij}(\tau)\pi_{i,\infty}\pi_{j,\infty}\right)} \\
&* \left[\begin{array}{l} \frac{1-\gamma}{\psi-1} - A_p\left(p_t-\overline{p}\right) - \sum_{i=1}^{m} A_i\left(\pi_{i,t}-\pi_{i,\infty}\right) - B_p\left(p_t^2-\overline{p}^2\right) - \sum_{i=1}^{m} B_i\left(\pi_{i,t}^2-\pi_{i,\infty}^2\right) \\ - \sum_{i=1}^{m} C_{pi}\left(p_t\pi_{i,t}-\overline{p}\pi_{i,\infty}\right) - \sum_{i<j} C_{ij}\left(\pi_{i,t}\pi_{j,t}-\pi_{i,\infty}\pi_{j,\infty}\right) \end{array}\right] \\
&- \frac{1-\gamma}{\psi-1}\beta\psi - A_0' - A_p' p_t - \sum_{i=1}^{m} A_i'\pi_{i,t} - B_p' p_t^2 - \sum_{i=1}^{m} B_i'\pi_{i,t}^2 - \sum_{i=1}^{m} C_{pi}' p_t \pi_{i,t} \\
&- \sum_{i<j} C_{ij}'\pi_{i,t}\pi_{j,t} + (1-\gamma) r + \left(A_p + 2B_p p_t + \sum_{i=1}^{m} C_{pi}\pi_{i,t}\right) * \kappa_p(\bar{\mathbb{P}}\pi_t - p_t) \\
&+ \frac{1}{2}\left[2B_p + \left(A_p + 2B_p p_t + \sum_{i=1}^{m} C_{pi}\pi_{i,t}\right)^2\right] \sigma_{\mathbf{p}}\sigma_{\mathbf{p}}^{\mathbf{T}} \\
&+ \sum_{i=1}^{m}\left[C_{pi} + \left(A_p + 2B_p p_t + \sum_{k=1}^{m} C_{pk}\pi_{k,t}\right)\left(A_i + 2B_i\pi_{i,t} + C_{pi}p_t + \sum_{k\neq i} C_{ik}\pi_{k,t}\right)\right] \sigma_{\mathbf{p}}\sigma_{\mathbf{i},\pi}^{\mathbf{T}} \\
&+ \frac{1-\gamma}{\gamma}\sum_{i=1}^{m}\left(A_p + 2B_p p_t + \sum_{k=1}^{m} C_{pk}\pi_{k,t}\right)\left(A_i + 2B_i\pi_{i,t} + C_{pi}p_t + \sum_{k\neq i} C_{ik}\pi_{k,t}\right) \overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{i},\pi}^{\mathbf{T}} \\
&+ \sum_{i,j=1}^{m}\lambda_{ji}\pi_{j,t}\left(A_i + 2B_i\pi_{i,t} + C_{pi}p_t + \sum_{k\neq i} C_{ik}\pi_{k,t}\right) + \sum_{i=1}^{m} B_i\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{i},\pi}^{\mathbf{T}} + \sum_{i<j} C_{ij}\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{j},\pi}^{\mathbf{T}} \\
&+ \frac{1}{2}\sum_{i,j=1}^{m}\left(A_i + 2B_i\pi_{i,t} + C_{pi}p_t + \sum_{k\neq i} C_{ik}\pi_{k,t}\right)\left(A_j + 2B_j\pi_{j,t} + C_{pj}p_t + \sum_{k\neq j} C_{jk}\pi_{k,t}\right) \sigma_{\mathbf{i},\pi}\sigma_{\mathbf{j},\pi}^{\mathbf{T}} \\
&+ \frac{1}{2}\sum_{i,j=1}^{m}\frac{1-\gamma}{\gamma}\left(A_i + 2B_i\pi_{i,t} + C_{pi}p_t + \sum_{k\neq i} C_{ik}\pi_{k,t}\right)\left(A_j + 2B_j\pi_{j,t} + C_{pj}p_t + \sum_{k\neq j} C_{jk}\pi_{k,t}\right) \overline{\sigma}_{\mathbf{i},\pi}\overline{\sigma}_{\mathbf{j},\pi}^{\mathbf{T}} \\
&+ \frac{1}{2}\frac{1-\gamma}{\gamma}\left(\mathbb{A}\pi_{\mathbf{t}} + \mathbb{B}p_t - \mathbf{r1}\right)^{\mathbf{T}}\left[\left(\mathbf{V}\pi_\infty\right)\left(\mathbf{V}\pi_\infty\right)^{\mathbf{T}}\right]^{-\mathbf{1}}\left(\mathbb{A}\pi_{\mathbf{t}} + \mathbb{B}p_t - \mathbf{r1}\right) \\
&+ \frac{1}{2}\frac{1-\gamma}{\gamma}\left(A_p + 2B_p p_t + \sum_{i=1}^{m} C_{pi}\pi_{i,t}\right)^2 \overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{p}}^{\mathbf{T}} \\
&+ \frac{1-\gamma}{\gamma}\left(A_p + 2B_p p_t + \sum_{i=1}^{m} C_{pi}\pi_{i,t}\right)\overline{\sigma}_{\mathbf{p}}\left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}}\left(\mathbb{A}\pi_{\mathbf{t}} + \mathbb{B}p_t - \mathbf{r1}\right) \\
&+ \frac{1-\gamma}{\gamma}\sum_{i=1}^{m}\left(A_i + 2B_i\pi_{i,t} + C_{pi}p_t + \sum_{k\neq i} C_{ik}\pi_{k,t}\right)\overline{\sigma}_{\mathbf{i},\pi}\left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}}\left(\mathbb{A}\pi_{\mathbf{t}} + \mathbb{B}p_t - \mathbf{r1}\right).
\end{aligned} \tag{16}
$$

As before, we match the coefficients and solve a system to find the time-varying coefficients of function $g\left(p_t, \pi_t, \tau\right)$. The system follows.

Equation a (independent term):

$$
\begin{aligned}
A_0' \; = \; & \beta^{\psi} e^{\frac{1-\psi}{1-\gamma}\left(A_0 + A_p\overline{p} + \sum_{i=1}^{m} A_i\pi_{i,\infty} + B_p\overline{p}^2 + \sum_{i=1}^{m} B_i\pi_{i,\infty}^2 + \sum_{i=1}^{m} C_{pi}\overline{p}\pi_{i,\infty} + \sum_{i<j} C_{ij}\pi_{i,\infty}\pi_{j,\infty}\right)} \\
& * \left(\frac{1-\gamma}{\psi-1} + A_p\overline{p} + \sum_{i=1}^{m} A_i\pi_{i,\infty} + B_p\overline{p}^2 + \sum_{i=1}^{m} B_i\pi_{i,\infty}^2 + \sum_{i=1}^{m} C_{pi}\overline{p}\pi_{i,\infty} + \sum_{i<j} C_{ij}\pi_{i,\infty}\pi_{j,\infty}\right) \\
& + (1-\gamma)\, r - \frac{1-\gamma}{\psi-1}\beta\psi + \left(B_p + \frac{1}{2}A_p^2\right)\sigma_{\mathbf{p}}\sigma_{\mathbf{p}}^{\mathbf{T}} + \frac{1-\gamma}{2\gamma}A_p^2\overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{p}}^{\mathbf{T}} + \sum_{i=1}^{m}\left(C_{pi} + A_pA_i\right)\sigma_{\mathbf{p}}\sigma_{\mathbf{i},\pi}^{\mathbf{T}} \\
& + \frac{1-\gamma}{\gamma}A_p\sum_{i=1}^{m} A_i\overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{i},\pi}^{\mathbf{T}} + \sum_{i=1}^{m} B_i\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{i},\pi}^{\mathbf{T}} + \sum_{i<j} C_{ij}\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{j},\pi}^{\mathbf{T}} + \frac{1}{2}\sum_{i,j=1}^{m} A_iA_j\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{j},\pi}^{\mathbf{T}} \\
& + \frac{1-\gamma}{2\gamma}\sum_{i,j=1}^{m} A_iA_j\overline{\sigma}_{\mathbf{i},\pi}\overline{\sigma}_{\mathbf{j},\pi}^{\mathbf{T}} + \frac{1-\gamma}{2\gamma}r^2\mathbf{1}^{\mathbf{T}}\left[\left(\mathbf{V}\pi_\infty\right)\left(\mathbf{V}\pi_\infty\right)^{\mathbf{T}}\right]^{-\mathbf{1}}\mathbf{1} - \frac{1-\gamma}{\gamma}rA_p\overline{\sigma}_{\mathbf{p}}\left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}}\mathbf{1} \\
& - \frac{1-\gamma}{\gamma}r\sum_{i=1}^{m} A_i\overline{\sigma}_{\mathbf{i},\pi}\left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}}\mathbf{1}.
\end{aligned}
\tag{17a}
$$

Equation b (p_t term):

$$
\begin{aligned}
A_p' \; = \; & -A_p\kappa_p - A_p\beta^{\psi} e^{\frac{1-\psi}{1-\gamma}\left(A_0 + A_p\overline{p} + \sum_{i=1}^{m} A_i\pi_{i,\infty} + B_p\overline{p}^2 + \sum_{i=1}^{m} B_i\pi_{i,\infty}^2 + \sum_{i=1}^{m} C_{pi}\overline{p}\pi_{i,\infty} + \sum_{i<j} C_{ij}\pi_{i,\infty}\pi_{j,\infty}\right)} \\
& + 2A_pB_p\sigma_{\mathbf{p}}\sigma_{\mathbf{p}}^{\mathbf{T}} + 2\frac{1-\gamma}{\gamma}A_pB_p\overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{p}}^{\mathbf{T}} + \sum_{i=1}^{m}\left(A_pC_{pi} + 2A_iB_p\right)\sigma_{\mathbf{p}}\sigma_{\mathbf{i},\pi}^{\mathbf{T}} \\
& + \frac{1-\gamma}{\gamma}\sum_{i=1}^{m}\left(A_pC_{pi} + 2A_iB_p\right)\overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{i},\pi}^{\mathbf{T}} + \frac{1}{2}\sum_{i,j=1}^{m}\left(A_iC_{pj} + A_jC_{pi}\right)\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{j},\pi}^{\mathbf{T}} \\
& + \frac{1-\gamma}{2\gamma}\sum_{i,j=1}^{m}\left(A_iC_{pj} + A_jC_{pi}\right)\overline{\sigma}_{\mathbf{i},\pi}\overline{\sigma}_{\mathbf{j},\pi}^{\mathbf{T}} - \frac{1-\gamma}{\gamma}r\mathbb{B}^{\mathbf{T}}\left[\left(\mathbf{V}\pi_\infty\right)\left(\mathbf{V}\pi_\infty\right)^{\mathbf{T}}\right]^{-\mathbf{1}}\mathbf{1} \\
& + \frac{1-\gamma}{\gamma}A_p\overline{\sigma}_{\mathbf{p}}\left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}}\mathbb{B} - 2\frac{1-\gamma}{\gamma}rB_p\overline{\sigma}_{\mathbf{p}}\left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}}\mathbf{1} \\
& + \frac{1-\gamma}{\gamma}\sum_{i=1}^{m} A_i\overline{\sigma}_{\mathbf{i},\pi}\left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}}\mathbb{B} - \frac{1-\gamma}{\gamma}r\sum_{i=1}^{m} C_{pi}\overline{\sigma}_{\mathbf{i},\pi}\left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}}\mathbf{1}.
\end{aligned}
\tag{17b}
$$

Equation c ($\pi_{i,t}$, $i \in R$, terms):

$$
\begin{aligned}
A_i' \;=\; & -A_i \beta^{\psi} e^{\frac{1-\psi}{1-\gamma}\left(A_0 + A_p \overline{p} + \sum_{i=1}^{m} A_i \pi_{i,\infty} + B_p \overline{p}^2 + \sum_{i=1}^{m} B_i \pi_{i,\infty}^2 + \sum_{i=1}^{m} C_{pi} \overline{p} \pi_{i,\infty} + \sum_{i<j} C_{ij} \pi_{i,\infty} \pi_{j,\infty}\right)} \\
& + A_p \kappa_p \bar{p}_i + A_p C_{pi} \sigma_{\mathbf{p}} \sigma_{\mathbf{p}}^{\mathbf{T}} + \frac{1-\gamma}{\gamma} A_p C_{pi} \overline{\sigma}_{\mathbf{p}} \overline{\sigma}_{\mathbf{p}}^{\mathbf{T}} + \left(2A_p B_i + A_i C_{pi}\right) \sigma_{\mathbf{p}} \sigma_{\mathbf{i},\pi}^{\mathbf{T}} \\
& + \sum_{j \neq i} \left(A_p C_{ji} + A_j C_{pi}\right) \sigma_{\mathbf{p}} \sigma_{\mathbf{j},\pi}^{\mathbf{T}} + \frac{1-\gamma}{\gamma} \left(2A_p B_i + A_i C_{pi}\right) \overline{\sigma}_{\mathbf{p}} \overline{\sigma}_{\mathbf{i},\pi}^{\mathbf{T}} + \\
& + \frac{1-\gamma}{\gamma} \sum_{j \neq i} \left(A_p C_{ji} + A_j C_{pi}\right) \overline{\sigma}_{\mathbf{p}} \overline{\sigma}_{\mathbf{j},\pi}^{\mathbf{T}} + \sum_{j=1}^{m} \lambda_{ij} A_j + 2A_i B_i \sigma_{\mathbf{i},\pi} \sigma_{\mathbf{i},\pi}^{\mathbf{T}} \\
& + \sum_{j \neq i} \left(A_i C_{ji} + 2A_j B_i\right) \sigma_{\mathbf{i},\pi} \sigma_{\mathbf{j},\pi}^{\mathbf{T}} + \frac{1}{2} \sum_{j,k \neq i} \left(A_j C_{ki} + A_k C_{ji}\right) \sigma_{\mathbf{j},\pi} \sigma_{\mathbf{k},\pi}^{\mathbf{T}} \\
& + 2\frac{1-\gamma}{\gamma} A_i B_i \overline{\sigma}_{\mathbf{i},\pi} \overline{\sigma}_{\mathbf{i},\pi}^{\mathbf{T}} + \frac{1-\gamma}{\gamma} \sum_{j \neq i} \left(A_i C_{ji} + 2A_j B_i\right) \overline{\sigma}_{\mathbf{i},\pi} \overline{\sigma}_{\mathbf{j},\pi}^{\mathbf{T}} \\
& + \frac{1-\gamma}{2\gamma} \sum_{j,k \neq i} \left(A_j C_{ki} + A_k C_{ji}\right) \overline{\sigma}_{\mathbf{j},\pi} \overline{\sigma}_{\mathbf{k},\pi}^{\mathbf{T}} - \frac{1-\gamma}{\gamma} r \mathbf{1}^{\mathbf{T}} \left[\left(\mathbf{V}\pi_{\infty}\right)\left(\mathbf{V}\pi_{\infty}\right)^{\mathbf{T}}\right]^{-1} \mathbb{A}_{:,i} \\
& + \frac{1-\gamma}{\gamma} \left[A_p \overline{\sigma}_{\mathbf{p}} \left(\mathbf{V}\pi_{\infty}\right)^{-\mathbf{1}} \mathbb{A}_{:,i} - r C_{pi} \overline{\sigma}_{\mathbf{p}} \left(\mathbf{V}\pi_{\infty}\right)^{-\mathbf{1}} \mathbf{1} + \sum_{j=1}^{m} A_j \overline{\sigma}_{\mathbf{j},\pi} \left(\mathbf{V}\pi_{\infty}\right)^{-\mathbf{1}} \mathbb{A}_{:,i}\right] \\
& - 2\frac{1-\gamma}{\gamma} r B_i \overline{\sigma}_{\mathbf{i},\pi} \left(\mathbf{V}\pi_{\infty}\right)^{-\mathbf{1}} \mathbf{1} - \frac{1-\gamma}{\gamma} r \sum_{j \neq i} C_{ji} \overline{\sigma}_{\mathbf{j},\pi} \left(\mathbf{V}\pi_{\infty}\right)^{-\mathbf{1}} \mathbf{1}.
\end{aligned}
\tag{17c}
$$

Equation d (p_t^2 term):

$$
\begin{aligned}
B_p' \;=\; & -2B_p \kappa_p - B_p \beta^{\psi} e^{\frac{1-\psi}{1-\gamma}\left(A_0 + A_p \overline{p} + \sum_{i=1}^{m} A_i \pi_{i,\infty} + B_p \overline{p}^2 + \sum_{i=1}^{m} B_i \pi_{i,\infty}^2 + \sum_{i=1}^{m} C_{pi} \overline{p} \pi_{i,\infty} + \sum_{i<j} C_{ij} \pi_{i,\infty} \pi_{j,\infty}\right)} \\
& + 2B_p^2 \sigma_{\mathbf{p}} \sigma_{\mathbf{p}}^{\mathbf{T}} + 2\sum_{i=1}^{m} B_p C_{pi} \sigma_{\mathbf{p}} \sigma_{\mathbf{i},\pi}^{\mathbf{T}} + 2\frac{1-\gamma}{\gamma} \sum_{i=1}^{m} B_p C_{pi} \overline{\sigma}_{\mathbf{p}} \overline{\sigma}_{\mathbf{i},\pi}^{\mathbf{T}} \\
& + \frac{1}{2} \sum_{i,j=1}^{m} C_{pi} C_{pj} \sigma_{\mathbf{i},\pi} \sigma_{\mathbf{j},\pi}^{\mathbf{T}} + \frac{1-\gamma}{2\gamma} \sum_{i,j=1}^{m} C_{pi} C_{pj} \overline{\sigma}_{\mathbf{i},\pi} \overline{\sigma}_{\mathbf{j},\pi}^{\mathbf{T}} \\
& + \frac{1-\gamma}{2\gamma} \mathbb{B}^{\mathbf{T}} \left[\left(\mathbf{V}\pi_{\infty}\right)\left(\mathbf{V}\pi_{\infty}\right)^{\mathbf{T}}\right]^{-1} \mathbb{B} + 2\frac{1-\gamma}{\gamma} B_p^2 \overline{\sigma}_{\mathbf{p}} \overline{\sigma}_{\mathbf{p}}^{\mathbf{T}} \\
& + 2\frac{1-\gamma}{\gamma} B_p \overline{\sigma}_{\mathbf{p}} \left(\mathbf{V}\pi_{\infty}\right)^{-\mathbf{1}} \mathbb{B} + \frac{1-\gamma}{\gamma} \sum_{i=1}^{m} C_{pi} \overline{\sigma}_{\mathbf{i},\pi} \left(\mathbf{V}\pi_{\infty}\right)^{-\mathbf{1}} \mathbb{B}.
\end{aligned}
\tag{17d}
$$

Equation e ($\pi^2_{i,t}$, $i \in R$, terms):

$$\begin{aligned}
B_i' &= -B_i\beta^{\psi} e^{\frac{1-\psi}{1-\gamma}\left(A_0+A_p\bar{p}+\sum_{i=1}^{m} A_i\pi_{i,\infty}+B_p\bar{p}^2+\sum_{i=1}^{m} B_i\pi^2_{i,\infty}+\sum_{i=1}^{m} C_{pi}\bar{p}\pi_{i,\infty}+\sum_{i<j} C_{ij}\pi_{i,\infty}\pi_{j,\infty}\right)} \\
&+ C_{pi}\kappa_p\bar{p}_i + \frac{1}{2}C^2_{pi}\sigma_{\mathbf{p}}\sigma^{\mathbf{T}}_{\mathbf{p}} + \frac{1-\gamma}{2\gamma}C^2_{pi}\overline{\sigma}_{\mathbf{p}}\overline{\sigma}^{\mathbf{T}}_{\mathbf{p}} + 2C_{pi}B_i\sigma_{\mathbf{p}}\sigma^{\mathbf{T}}_{\mathbf{i},\pi} \\
&+ 2\frac{1-\gamma}{\gamma}C_{pi}B_i\overline{\sigma}_{\mathbf{p}}\overline{\sigma}^{\mathbf{T}}_{\mathbf{i},\pi} + \sum_{j\neq i} C_{pi}C_{ji}\sigma_{\mathbf{p}}\sigma^{\mathbf{T}}_{\mathbf{j},\pi} + \frac{1-\gamma}{\gamma}\sum_{j\neq i} C_{pi}C_{ji}\overline{\sigma}_{\mathbf{p}}\overline{\sigma}^{\mathbf{T}}_{\mathbf{j},\pi} \\
&+ 2\lambda_{ii}B_i + \sum_{j\neq i}\lambda_{ij}C_{ji} + 2B_i^2\sigma_{\mathbf{i},\pi}\sigma^{\mathbf{T}}_{\mathbf{i},\pi} + 2\frac{1-\gamma}{\gamma}B_i^2\overline{\sigma}_{\mathbf{i},\pi}\overline{\sigma}^{\mathbf{T}}_{\mathbf{i},\pi} \\
&+ 2B_i\sum_{j\neq i} C_{ji}\sigma_{\mathbf{i},\pi}\sigma^{\mathbf{T}}_{\mathbf{j},\pi} + 2\frac{1-\gamma}{\gamma}B_i\sum_{j\neq i} C_{ji}\overline{\sigma}_{\mathbf{i},\pi}\overline{\sigma}^{\mathbf{T}}_{\mathbf{j},\pi} + \frac{1}{2}\sum_{j,k\neq i} C_{ji}C_{ki}\sigma_{\mathbf{j},\pi}\sigma^{\mathbf{T}}_{\mathbf{k},\pi} \\
&+ \frac{1-\gamma}{2\gamma}\sum_{j,k\neq i} C_{ji}C_{ki}\overline{\sigma}_{\mathbf{j},\pi}\overline{\sigma}^{\mathbf{T}}_{\mathbf{k},\pi} + \frac{1-\gamma}{2\gamma}\mathbb{A}^{\mathbf{T}}_{:,i}\left[(\mathbf{V}\pi_\infty)(\mathbf{V}\pi_\infty)^{\mathbf{T}}\right]^{-1}\mathbb{A}_{:,i} \\
&+ \frac{1-\gamma}{\gamma}C_{pi}\overline{\sigma}_{\mathbf{p}}(\mathbf{V}\pi_\infty)^{-1}\mathbb{A}_{:,i} + 2\frac{1-\gamma}{\gamma}B_i\overline{\sigma}_{\mathbf{i},\pi}(\mathbf{V}\pi_\infty)^{-1}\mathbb{A}_{:,i} \\
&+ \frac{1-\gamma}{\gamma}\sum_{j\neq i} C_{ji}\overline{\sigma}_{\mathbf{j},\pi}(\mathbf{V}\pi_\infty)^{-1}\mathbb{A}_{:,i}.
\end{aligned} \tag{17e}$$

Equation f ($p_t\pi_{i,t}$, $i \in R$, terms):

$$\begin{aligned}
C_{pi}' &= -C_{pi}\beta^{\psi} e^{\frac{1-\psi}{1-\gamma}\left(A_0+A_p\bar{p}+\sum_{i=1}^{m} A_i\pi_{i,\infty}+B_p\bar{p}^2+\sum_{i=1}^{m} B_i\pi^2_{i,\infty}+\sum_{i=1}^{m} C_{pi}\bar{p}\pi_{i,\infty}+\sum_{i<j} C_{ij}\pi_{i,\infty}\pi_{j,\infty}\right)} \\
&+ 2B_p\kappa_p\bar{p}_i - C_{pi}\kappa_p + 2B_pC_{pi}\sigma_{\mathbf{p}}\sigma^{\mathbf{T}}_{\mathbf{p}} + 2\frac{1-\gamma}{\gamma}B_pC_{pi}\overline{\sigma}_{\mathbf{p}}\overline{\sigma}^{\mathbf{T}}_{\mathbf{p}} + 4B_pB_i\sigma_{\mathbf{p}}\sigma^{\mathbf{T}}_{\mathbf{i},\pi} + C^2_{pi}\sigma_{\mathbf{p}}\sigma^{\mathbf{T}}_{\mathbf{i},\pi} \\
&+ \sum_{j\neq i}(2B_pC_{ji} + C_{pi}C_{pj})\sigma_{\mathbf{p}}\sigma^{\mathbf{T}}_{\mathbf{j},\pi} + 4\frac{1-\gamma}{\gamma}B_pB_i\overline{\sigma}_{\mathbf{p}}\overline{\sigma}^{\mathbf{T}}_{\mathbf{i},\pi} + \frac{1-\gamma}{\gamma}C^2_{pi}\overline{\sigma}_{\mathbf{p}}\overline{\sigma}^{\mathbf{T}}_{\mathbf{i},\pi} \\
&+ \frac{1-\gamma}{\gamma}\sum_{j\neq i}(2B_pC_{ji} + C_{pi}C_{pj})\overline{\sigma}_{\mathbf{p}}\overline{\sigma}^{\mathbf{T}}_{\mathbf{j},\pi} + \sum_{j=1}^{m}\lambda_{ij}C_{pj} + 2B_iC_{pi}\sigma_{\mathbf{i},\pi}\sigma^{\mathbf{T}}_{\mathbf{i},\pi} \\
&+ 2\frac{1-\gamma}{\gamma}B_iC_{pi}\overline{\sigma}_{\mathbf{i},\pi}\overline{\sigma}^{\mathbf{T}}_{\mathbf{i},\pi} + \sum_{j\neq i}(2B_iC_{pj} + C_{pi}C_{ji})\sigma_{\mathbf{i},\pi}\sigma^{\mathbf{T}}_{\mathbf{j},\pi} \\
&+ \frac{1-\gamma}{\gamma}\sum_{j\neq i}(2B_iC_{pj} + C_{pi}C_{ji})\overline{\sigma}_{\mathbf{i},\pi}\overline{\sigma}^{\mathbf{T}}_{\mathbf{j},\pi} + \frac{1}{2}\sum_{j,k\neq i}(C_{pj}C_{ki} + C_{pk}C_{ji})\sigma_{\mathbf{j},\pi}\sigma^{\mathbf{T}}_{\mathbf{k},\pi} \\
&\frac{1-\gamma}{2\gamma}\sum_{j,k\neq i}(C_{pj}C_{ki} + C_{pk}C_{ji})\overline{\sigma}_{\mathbf{j},\pi}\overline{\sigma}^{\mathbf{T}}_{\mathbf{k},\pi} + \frac{1-\gamma}{\gamma}(\mathbb{A}_{:,i})^{\mathbf{T}}\left[(\mathbf{V}\pi_\infty)(\mathbf{V}\pi_\infty)^{\mathbf{T}}\right]^{-1}\mathbb{B} \\
&+ \frac{1-\gamma}{\gamma}\left[2B_p\overline{\sigma}_{\mathbf{p}}(\mathbf{V}\pi_\infty)^{-1}\mathbb{A}_{:,i} + C_{pi}\overline{\sigma}_{\mathbf{p}}(\mathbf{V}\pi_\infty)^{-1}\mathbb{B} + \sum_{j=1}^{m} C_{pj}\overline{\sigma}_{\mathbf{j},\pi}(\mathbf{V}\pi_\infty)^{-1}\mathbb{A}_{:,i}\right] \\
&+ 2\frac{1-\gamma}{\gamma}B_i\overline{\sigma}_{\mathbf{i},\pi}(\mathbf{V}\pi_\infty)^{-1}\mathbb{B} + \frac{1-\gamma}{\gamma}\sum_{j\neq i} C_{ji}\overline{\sigma}_{\mathbf{j},\pi}(\mathbf{V}\pi_\infty)^{-1}\mathbb{B}.
\end{aligned} \tag{17f}$$

Equation g ($\pi_{i,t}\pi_{j,t}$, $i<j\in R$, terms):

$$
\begin{aligned}
C'_{ij} \;=\; & -C_{ij}\beta^{\psi}e^{\frac{1-\psi}{1-\gamma}\left(A_0+A_p\bar{p}+\sum_{i=1}^{m}A_i\pi_{i,\infty}+B_p\bar{p}^2+\sum_{i=1}^{m}B_i\pi_{i,\infty}^2+\sum_{i=1}^{m}C_{pi}\bar{p}\pi_{i,\infty}+\sum_{i<j}C_{ij}\pi_{i,\infty}\pi_{j,\infty}\right)} \\
& + C_{pi}\kappa_p\bar{p}_j + C_{pj}\kappa_p\bar{p}_i + C_{pi}C_{pj}\sigma_{\mathbf{p}}\sigma_{\mathbf{p}}^{\mathbf{T}} + \frac{1-\gamma}{\gamma}C_{pi}C_{pj}\sigma_{\mathbf{p}}\overline{\sigma}_{\mathbf{p}}^{\mathbf{T}} \\
& + \left(2B_iC_{pj}+C_{pi}C_{ij}\right)\sigma_{\mathbf{p}}\sigma_{\mathbf{i},\pi}^{\mathbf{T}} + \left(2B_jC_{pi}+C_{pj}C_{ij}\right)\sigma_{\mathbf{p}}\sigma_{\mathbf{j},\pi}^{\mathbf{T}} \\
& + \frac{1-\gamma}{\gamma}\left[\left(2B_iC_{pj}+C_{pi}C_{ij}\right)\overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{i},\pi}^{\mathbf{T}} + \left(2B_jC_{pi}+C_{pj}C_{ij}\right)\overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{j},\pi}^{\mathbf{T}}\right] \\
& + \sum_{k\neq i,j}\left(C_{pi}C_{kj}+C_{pj}C_{ki}\right)\sigma_{\mathbf{p}}\sigma_{\mathbf{k},\pi}^{\mathbf{T}} + \frac{1-\gamma}{\gamma}\sum_{k\neq i,j}\left(C_{pi}C_{kj}+C_{pj}C_{ki}\right)\overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{k},\pi}^{\mathbf{T}} \\
& + 2\lambda_{ij}B_j + 2\lambda_{ji}B_i + \lambda_{ii}C_{ij} + \lambda_{jj}C_{ij} + \sum_{k\neq i,j}\left(\lambda_{ik}C_{kj}+\lambda_{jk}C_{ki}\right) \\
& + 2B_iC_{ij}\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{i},\pi}^{\mathbf{T}} + 2B_jC_{ij}\sigma_{\mathbf{j},\pi}\sigma_{\mathbf{j},\pi}^{\mathbf{T}} + 2\frac{1-\gamma}{\gamma}\left[B_iC_{ij}\overline{\sigma}_{\mathbf{i},\pi}\overline{\sigma}_{\mathbf{i},\pi}^{\mathbf{T}} + B_jC_{ij}\overline{\sigma}_{\mathbf{j},\pi}\overline{\sigma}_{\mathbf{j},\pi}^{\mathbf{T}}\right] \\
& + \left(4B_iB_j+C_{ij}^2\right)\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{j},\pi}^{\mathbf{T}} + \frac{1-\gamma}{\gamma}\left(4B_iB_j+C_{ij}^2\right)\overline{\sigma}_{\mathbf{i},\pi}\overline{\sigma}_{\mathbf{j},\pi}^{\mathbf{T}} \\
& + \sum_{k\neq i,j}\left(2B_iC_{jk}+C_{ij}C_{ki}\right)\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{k},\pi}^{\mathbf{T}} + \sum_{k\neq i,j}\left(2B_jC_{ik}+C_{ji}C_{kj}\right)\sigma_{\mathbf{j},\pi}\sigma_{\mathbf{k},\pi}^{\mathbf{T}} \\
& + \frac{1-\gamma}{\gamma}\left[\sum_{k\neq i,j}\left(2B_iC_{jk}+C_{ij}C_{ki}\right)\overline{\sigma}_{\mathbf{i},\pi}\overline{\sigma}_{\mathbf{k},\pi}^{\mathbf{T}} + \sum_{k\neq i,j}\left(2B_jC_{ik}+C_{ji}C_{kj}\right)\overline{\sigma}_{\mathbf{j},\pi}\overline{\sigma}_{\mathbf{k},\pi}^{\mathbf{T}}\right] \\
& + \sum_{k<l\neq i,j}\left(C_{ki}C_{lj}+C_{kj}C_{li}\right)\sigma_{\mathbf{k},\pi}\sigma_{\mathbf{l},\pi}^{\mathbf{T}} + \sum_{k\neq i,j}C_{ik}C_{jk}\sigma_{\mathbf{k},\pi}\sigma_{\mathbf{k},\pi}^{\mathbf{T}} \\
& + \frac{1-\gamma}{\gamma}\left[\sum_{k<l\neq i,j}\left(C_{ki}C_{lj}+C_{kj}C_{li}\right)\overline{\sigma}_{\mathbf{k},\pi}\overline{\sigma}_{\mathbf{l},\pi}^{\mathbf{T}} + \sum_{k\neq i,j}C_{ik}C_{jk}\overline{\sigma}_{\mathbf{k},\pi}\overline{\sigma}_{\mathbf{k},\pi}^{\mathbf{T}}\right] \\
& + \frac{1-\gamma}{\gamma}\left(\mathbb{A}_{:,i}\right)^{\mathbf{T}}\left[\left(\mathbf{V}\pi_\infty\right)\left(\mathbf{V}\pi_\infty\right)^{\mathbf{T}}\right]^{-1}\mathbb{A}_{:,j} \\
& + \frac{1-\gamma}{\gamma}C_{pi}\overline{\sigma}_{\mathbf{p}}\left(\mathbf{V}\pi_\infty\right)^{-1}\mathbb{A}_{:,j} + \frac{1-\gamma}{\gamma}C_{pj}\overline{\sigma}_{\mathbf{p}}\left(\mathbf{V}\pi_\infty\right)^{-1}\mathbb{A}_{:,i} \\
& + 2\frac{1-\gamma}{\gamma}B_i\overline{\sigma}_{\mathbf{i},\pi}\left(\mathbf{V}\pi_\infty\right)^{-1}\mathbb{A}_{:,j} + \frac{1-\gamma}{\gamma}\sum_{k\neq i}C_{ik}\overline{\sigma}_{\mathbf{k},\pi}\left(\mathbf{V}\pi_\infty\right)^{-1}\mathbb{A}_{:,j} \\
& + 2\frac{1-\gamma}{\gamma}B_j\overline{\sigma}_{\mathbf{j},\pi}\left(\mathbf{V}\pi_\infty\right)^{-1}\mathbb{A}_{:,i} + \frac{1-\gamma}{\gamma}\sum_{k\neq j}C_{jk}\overline{\sigma}_{\mathbf{k},\pi}\left(\mathbf{V}\pi_\infty\right)^{-1}\mathbb{A}_{:,i}.
\end{aligned}
\tag{17g}
$$

5 The System of Equations to the Problem with Myopic Portfolio Strategy (No Consumption)

Equation a (independent term):

$$\begin{aligned}A_0' \;=\;& -\beta+(1-\gamma)\,r+\left(B_p+\frac{1}{2}A_p^2\right)\sigma_{\mathbf{p}}\sigma_{\mathbf{p}}^{\mathbf{T}}+\sum_{i=1}^{m}\left(C_{pi}+A_pA_i\right)\sigma_{\mathbf{p}}\sigma_{\mathbf{i},\pi}^{\mathbf{T}}\\&+\sum_{i=1}^{m}B_i\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{i},\pi}^{\mathbf{T}}+\sum_{i<j}C_{ij}\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{j},\pi}^{\mathbf{T}}+\frac{1}{2}\sum_{i,j=1}^{m}A_iA_j\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{j},\pi}^{\mathbf{T}}\\&+\frac{1-\gamma}{2\gamma}r^2\mathbf{1}^{\mathbf{T}}\left[\left(\mathbf{V}\pi_\infty\right)\left(\mathbf{V}\pi_\infty\right)^{\mathbf{T}}\right]^{-1}\mathbf{1}\\&-\frac{1-\gamma}{\gamma}rA_p\overline{\sigma}_{\mathbf{p}}\left(\mathbf{V}\pi_\infty\right)^{-1}\mathbf{1}-\frac{1-\gamma}{\gamma}r\sum_{i=1}^{m}A_i\overline{\sigma}_{\mathbf{i},\pi}\left(\mathbf{V}\pi_\infty\right)^{-1}\mathbf{1}.\end{aligned}\tag{18a}$$

Equation b (p_t term):

$$\begin{aligned}A_p' \;=\;& -A_p\kappa_p+2A_pB_p\sigma_{\mathbf{p}}\sigma_{\mathbf{p}}^{\mathbf{T}}+\sum_{i=1}^{m}\left(A_pC_{pi}+2A_iB_p\right)\sigma_{\mathbf{p}}\sigma_{\mathbf{i},\pi}^{\mathbf{T}}\\&+\frac{1}{2}\sum_{i,j=1}^{m}\left(A_iC_{pj}+A_jC_{pi}\right)\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{j},\pi}^{\mathbf{T}}-\frac{1-\gamma}{\gamma}r\mathbb{B}^{\mathbf{T}}\left[\left(\mathbf{V}\pi_\infty\right)\left(\mathbf{V}\pi_\infty\right)^{\mathbf{T}}\right]^{-1}\mathbf{1}\\&+\frac{1-\gamma}{\gamma}A_p\overline{\sigma}_{\mathbf{p}}\left(\mathbf{V}\pi_\infty\right)^{-1}\mathbb{B}-2\frac{1-\gamma}{\gamma}rB_p\overline{\sigma}_{\mathbf{p}}\left(\mathbf{V}\pi_\infty\right)^{-1}\mathbf{1}\\&+\frac{1-\gamma}{\gamma}\sum_{i=1}^{m}A_i\overline{\sigma}_{\mathbf{i},\pi}\left(\mathbf{V}\pi_\infty\right)^{-1}\mathbb{B}-\frac{1-\gamma}{\gamma}r\sum_{i=1}^{m}C_{pi}\overline{\sigma}_{\mathbf{i},\pi}\left(\mathbf{V}\pi_\infty\right)^{-1}\mathbf{1}.\end{aligned}\tag{18b}$$

Equation c ($\pi_{i,t}$, $i\in R$, terms):

$$\begin{aligned}A_i' \;=\;& A_p\kappa_p\bar{p}_i+A_pC_{pi}\sigma_{\mathbf{p}}\sigma_{\mathbf{p}}^{\mathbf{T}}+\left(2A_pB_i+A_iC_{pi}\right)\sigma_{\mathbf{p}}\sigma_{\mathbf{i},\pi}^{\mathbf{T}}+\sum_{j\neq i}\left(A_pC_{ji}+A_jC_{pi}\right)\sigma_{\mathbf{p}}\sigma_{\mathbf{j},\pi}^{\mathbf{T}}\\&+\sum_{j=1}^{m}\lambda_{ij}A_j+2A_iB_i\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{i},\pi}^{\mathbf{T}}+\sum_{j\neq i}\left(A_iC_{ji}+2A_jB_i\right)\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{j},\pi}^{\mathbf{T}}\\&+\frac{1}{2}\sum_{j,k\neq i}\left(A_jC_{ki}+A_kC_{ji}\right)\sigma_{\mathbf{j},\pi}\sigma_{\mathbf{k},\pi}^{\mathbf{T}}-\frac{1-\gamma}{\gamma}r\mathbf{1}^{\mathbf{T}}\left[\left(\mathbf{V}\pi_\infty\right)\left(\mathbf{V}\pi_\infty\right)^{\mathbf{T}}\right]^{-1}\mathbb{A}_{:,i}\\&+\frac{1-\gamma}{\gamma}\left[A_p\overline{\sigma}_{\mathbf{p}}\left(\mathbf{V}\pi_\infty\right)^{-1}\mathbb{A}_{:,i}-rC_{pi}\overline{\sigma}_{\mathbf{p}}\left(\mathbf{V}\pi_\infty\right)^{-1}\mathbf{1}+\sum_{j=1}^{m}A_j\overline{\sigma}_{\mathbf{j},\pi}\left(\mathbf{V}\pi_\infty\right)^{-1}\mathbb{A}_{:,i}\right]\\&-2\frac{1-\gamma}{\gamma}rB_i\overline{\sigma}_{\mathbf{i},\pi}\left(\mathbf{V}\pi_\infty\right)^{-1}\mathbf{1}-\frac{1-\gamma}{\gamma}r\sum_{j\neq i}C_{ji}\overline{\sigma}_{\mathbf{j},\pi}\left(\mathbf{V}\pi_\infty\right)^{-1}\mathbf{1}.\end{aligned}\tag{18c}$$

Equation d (p_t^2 term):

$$\begin{aligned}
B_p' &= -2B_p\kappa_p + 2B_p^2\sigma_{\mathbf{p}}\sigma_{\mathbf{p}}^{\mathbf{T}} + 2\sum_{i=1}^{m} B_p C_{pi}\sigma_{\mathbf{p}}\sigma_{\mathbf{i},\pi}^{\mathbf{T}} + \frac{1}{2}\sum_{i,j=1}^{m} C_{pi}C_{pj}\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{j},\pi}^{\mathbf{T}} \\
&+ \frac{1-\gamma}{2\gamma}\mathbb{B}^{\mathbf{T}}\left[(\mathbf{V}\pi_\infty)(\mathbf{V}\pi_\infty)^{\mathbf{T}}\right]^{-1}\mathbb{B} \\
&+ 2\frac{1-\gamma}{\gamma}B_p\overline{\sigma}_{\mathbf{p}}(\mathbf{V}\pi_\infty)^{-1}\mathbb{B} + \frac{1-\gamma}{\gamma}\sum_{i=1}^{m} C_{pi}\overline{\sigma}_{\mathbf{i},\pi}(\mathbf{V}\pi_\infty)^{-1}\mathbb{B}.
\end{aligned} \tag{18d}$$

Equation e ($\pi_{i,t}^2$, $i \in R$, terms):

$$\begin{aligned}
B_i' &= C_{pi}\kappa_p\bar{p}_i + \frac{1}{2}C_{pi}^2\sigma_{\mathbf{p}}\sigma_{\mathbf{p}}^{\mathbf{T}} + 2C_{pi}B_i\sigma_{\mathbf{p}}\sigma_{\mathbf{i},\pi}^{\mathbf{T}} + \sum_{j\neq i} C_{pi}C_{ji}\sigma_{\mathbf{p}}\sigma_{\mathbf{j},\pi}^{\mathbf{T}} + 2\lambda_{ii}B_i \\
&+ \sum_{j\neq i}\lambda_{ij}C_{ji} + 2B_i^2\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{i},\pi}^{\mathbf{T}} + 2B_i\sum_{j\neq i}C_{ji}\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{j},\pi}^{\mathbf{T}} \\
&+ \frac{1}{2}\sum_{j,k\neq i}C_{ji}C_{ki}\sigma_{\mathbf{j},\pi}\sigma_{\mathbf{k},\pi}^{\mathbf{T}} + \frac{1-\gamma}{2\gamma}\mathbb{A}_{:,i}^{\mathbf{T}}\left[(\mathbf{V}\pi_\infty)(\mathbf{V}\pi_\infty)^{\mathbf{T}}\right]^{-1}\mathbb{A}_{:,i} \\
&+ \frac{1-\gamma}{\gamma}C_{pi}\overline{\sigma}_{\mathbf{p}}(\mathbf{V}\pi_\infty)^{-1}\mathbb{A}_{:,i} + 2\frac{1-\gamma}{\gamma}B_i\overline{\sigma}_{\mathbf{i},\pi}(\mathbf{V}\pi_\infty)^{-1}\mathbb{A}_{:,i} \\
&+ \frac{1-\gamma}{\gamma}\sum_{j\neq i}C_{ji}\overline{\sigma}_{\mathbf{j},\pi}(\mathbf{V}\pi_\infty)^{-1}\mathbb{A}_{:,i}.
\end{aligned} \tag{18e}$$

Equation f ($p_t\pi_{i,t}$, $i \in R$, terms):

$$\begin{aligned}
C_{pi}' &= 2B_p\kappa_p\bar{p}_i - C_{pi}\kappa_p + 2B_pC_{pi}\sigma_{\mathbf{p}}\sigma_{\mathbf{p}}^{\mathbf{T}} + 4B_pB_i\sigma_{\mathbf{p}}\sigma_{\mathbf{i},\pi}^{\mathbf{T}} + C_{pi}^2\sigma_{\mathbf{p}}\sigma_{\mathbf{i},\pi}^{\mathbf{T}} \\
&+ \sum_{j\neq i}(2B_pC_{ji} + C_{pi}C_{pj})\sigma_{\mathbf{p}}\sigma_{\mathbf{j},\pi}^{\mathbf{T}} + \sum_{j=1}^{m}\lambda_{ij}C_{pj} \\
&+ 2B_iC_{pi}\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{i},\pi}^{\mathbf{T}} + \sum_{j\neq i}(2B_iC_{pj} + C_{pi}C_{ji})\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{j},\pi}^{\mathbf{T}} \\
&+ \frac{1}{2}\sum_{j,k\neq i}(C_{pj}C_{ki} + C_{pk}C_{ji})\sigma_{\mathbf{j},\pi}\sigma_{\mathbf{k},\pi}^{\mathbf{T}} + \frac{1-\gamma}{\gamma}(\mathbb{A}_{:,i})^{\mathbf{T}}\left[(\mathbf{V}\pi_\infty)(\mathbf{V}\pi_\infty)^{\mathbf{T}}\right]^{-1}\mathbb{B} \\
&+ \frac{1-\gamma}{\gamma}\left[2B_p\overline{\sigma}_{\mathbf{p}}(\mathbf{V}\pi_\infty)^{-1}\mathbb{A}_{:,i} + C_{pi}\overline{\sigma}_{\mathbf{p}}(\mathbf{V}\pi_\infty)^{-1}\mathbb{B} + \sum_{j=1}^{m}C_{pj}\overline{\sigma}_{\mathbf{j},\pi}(\mathbf{V}\pi_\infty)^{-1}\mathbb{A}_{:,i}\right] \\
&+ 2\frac{1-\gamma}{\gamma}B_i\overline{\sigma}_{\mathbf{i},\pi}(\mathbf{V}\pi_\infty)^{-1}\mathbb{B} + \frac{1-\gamma}{\gamma}\sum_{j\neq i}C_{ji}\overline{\sigma}_{\mathbf{j},\pi}(\mathbf{V}\pi_\infty)^{-1}\mathbb{B}.
\end{aligned} \tag{18f}$$

Equation g ($\pi_{i,t}\pi_{j,t}$, $i < j \in R$, terms):

$$
\begin{aligned}
C'_{ij} \;=\; & C_{pi}\kappa_p\bar{p}_j + C_{pj}\kappa_p\bar{p}_i + C_{pi}C_{pj}\sigma_{\mathbf{p}}\sigma_{\mathbf{p}}^{\mathbf{T}} + \left(2B_iC_{pj} + C_{pi}C_{ij}\right)\sigma_{\mathbf{p}}\sigma_{\mathbf{i},\pi}^{\mathbf{T}} \\
& + \left(2B_jC_{pi} + C_{pj}C_{ij}\right)\sigma_{\mathbf{p}}\sigma_{\mathbf{j},\pi}^{\mathbf{T}} + \sum_{k\neq i,j}\left(C_{pi}C_{kj} + C_{pj}C_{ki}\right)\sigma_{\mathbf{p}}\sigma_{\mathbf{k},\pi}^{\mathbf{T}} \\
& + 2\lambda_{ij}B_j + 2\lambda_{ji}B_i + \lambda_{ii}C_{ij} + \lambda_{jj}C_{ij} + \sum_{k\neq i,j}\left(\lambda_{ik}C_{kj} + \lambda_{jk}C_{ki}\right) \\
& + 2B_iC_{ij}\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{i},\pi}^{\mathbf{T}} + 2B_jC_{ij}\sigma_{\mathbf{j},\pi}\sigma_{\mathbf{j},\pi}^{\mathbf{T}} + \left(4B_iB_j + C_{ij}^2\right)\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{j},\pi}^{\mathbf{T}} \\
& + \sum_{k\neq i,j}\left(2B_iC_{jk} + C_{ij}C_{ki}\right)\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{k},\pi}^{\mathbf{T}} + \sum_{k\neq i,j}\left(2B_jC_{ik} + C_{ji}C_{kj}\right)\sigma_{\mathbf{j},\pi}\sigma_{\mathbf{k},\pi}^{\mathbf{T}} \\
& + \sum_{k<l\neq i,j}\left(C_{ki}C_{lj} + C_{kj}C_{li}\right)\sigma_{\mathbf{k},\pi}\sigma_{\mathbf{l},\pi}^{\mathbf{T}} + \sum_{k\neq i,j} C_{ik}C_{jk}\sigma_{\mathbf{k},\pi}\sigma_{\mathbf{k},\pi}^{\mathbf{T}} \\
& + \frac{1-\gamma}{\gamma}\left(\mathbb{A}_{:,i}\right)^{\mathbf{T}}\left[\left(\mathbf{V}\pi_\infty\right)\left(\mathbf{V}\pi_\infty\right)^{\mathbf{T}}\right]^{-\mathbf{1}}\mathbb{A}_{:,j} \\
& + \frac{1-\gamma}{\gamma}C_{pi}\overline{\sigma}_{\mathbf{p}}\left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}}\mathbb{A}_{:,j} + \frac{1-\gamma}{\gamma}C_{pj}\overline{\sigma}_{\mathbf{p}}\left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}}\mathbb{A}_{:,i} \\
& + 2\frac{1-\gamma}{\gamma}B_i\overline{\sigma}_{\mathbf{i},\pi}\left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}}\mathbb{A}_{:,j} + \frac{1-\gamma}{\gamma}\sum_{k\neq i} C_{ik}\overline{\sigma}_{\mathbf{k},\pi}\left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}}\mathbb{A}_{:,j} \\
& + 2\frac{1-\gamma}{\gamma}B_j\overline{\sigma}_{\mathbf{j},\pi}\left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}}\mathbb{A}_{:,i} + \frac{1-\gamma}{\gamma}\sum_{k\neq j} C_{jk}\overline{\sigma}_{\mathbf{k},\pi}\left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}}\mathbb{A}_{:,i}.
\end{aligned}
\tag{18g}
$$

6 Performance Measurement for a Power Utility Investor

To compare the performance of the various allocation strategies, we assume that investors preferences follow a power utility function:

$$U(R_i) = \frac{R_i^{1-\gamma}}{1-\gamma}, \tag{19}$$

where R_i denotes portfolio i's return and γ corresponds to the investor's coefficient of relative risk aversion. They maximize the expected utility of their portfolio returns at the horizon (T).

Instead of comparing the expected utility of the strategy returns, which is difficult to interpret over different horizons and levels of risk aversion, we follow Fugazza, Guidolin and Nicodano (2015) and compute the annualized certainty-equivalent return (CER):

$$u(W_t(1+(T/12)) \times CER(\hat{\alpha}_{i,t}(\gamma, T))) = E[u(W_{t+T}(\hat{\alpha}_{i,t}(\gamma, T)))], \tag{20a}$$

where $E[u(W_{t+T}(\hat{\omega}_{i,t}(\gamma, T)))]$ is the terminal wealth generated by the strategy returns $\hat{\omega}_{i,t}$, given the utility function $u(\cdot)$ and the investor's wealth W_t. Using the power utility function, we obtain the annualized out-of-sample CER for strategy i:

$$CER_i(\gamma, T) \equiv \frac{12}{T}\left\{\frac{1}{W_t}\left[\frac{1}{K-T}\sum_{\tau=1}^{K-T}[W_{\tau+T}(\hat{\omega}_{i,t}(\gamma, T))]^{1-\gamma}\right]^{\frac{1}{1-\gamma}} - 1\right\}, \tag{20b}$$

where K is the number of generated out-of-sample returns. In our application we compute this annualized CER for the multi-regime, single-regime and equal-weight strategies over three samples starting in 1960, 1980 and 2000 and ending on 2018.

7 An Application with Value and Growth Portfolios

The set of assets is made up of one risk-free asset and $n = 4$ risky assets: a growth index, a neutral index, a value index and a long-term bond index. The stocks indices were respectively selected based on the first, third and fifth quintiles of portfolios formed on book-to-market using NYSE, AMEX and NASDAQ firms.[6] The time-series of long-term bonds and the riskless asset are the same as in our application in the main text with large and small stock portfolios. The parameter estimates are reported in Table 2. Panel A shows the single regime parameters, while

[6]The stock indices were downloaded from Kenneth French's website.

the four-state parameters are in Panel B. The highest returns are obtained for the value firms. It should also be noted that neutral firms dominate growth firms in the single state and in all regimes, *i.e.*, they have higher returns and a slightly lower volatility. In the crash and recovery regimes, the returns are less extreme than with the size portfolios in the former application.

The portfolio shares are reported in Table 3. The positions are less extreme than in the former application with size portfolios, except in the bull regime. With ergodic probabilities at play, the investor borrows at the risk-free rate and shorts the growth portfolio to invest about equally in the neutral, value and bond portfolios, with little variation over the various horizons.

The out-of-sample returns statistics reported in Table 4 are less variable and leptokurtic than in the size-portfolio application in the main text. For the three samples, the CER difference favors the multi-regime model over the single-regime and equal-weight strategies but this difference is not statistically significant according to the bootstrap confidence intervals except for the equal-weight strategy in the 2000 sample.

8 Figures

Figure 1: Four-Regime Model with Four Assets Without Predictability: Filtered probabilities of being in each of the states. For every point in time, these probabilities are estimated using only the information available before time t. We begin our filter in January 1952. The data sample is from July 1962 to July 2018.

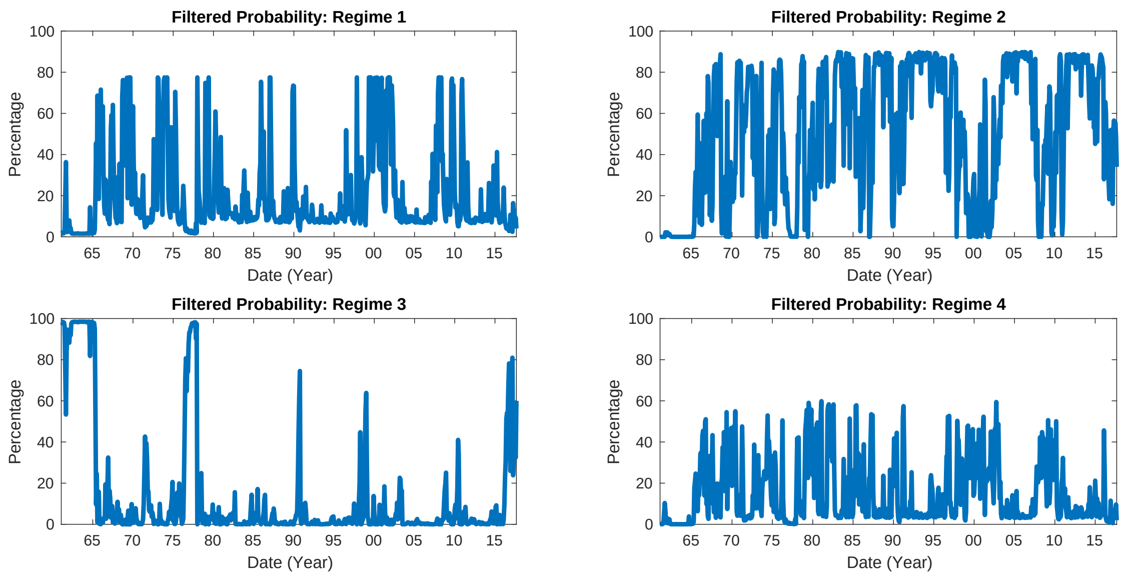


Figure 2: Four-Regime Model with Four Assets Without Predictability: Dynamic strategies for different horizons when the current regime is known. Short-selling is allowed. The horizon is measured in years. Investor's relative risk aversion is set $\gamma = 5$.

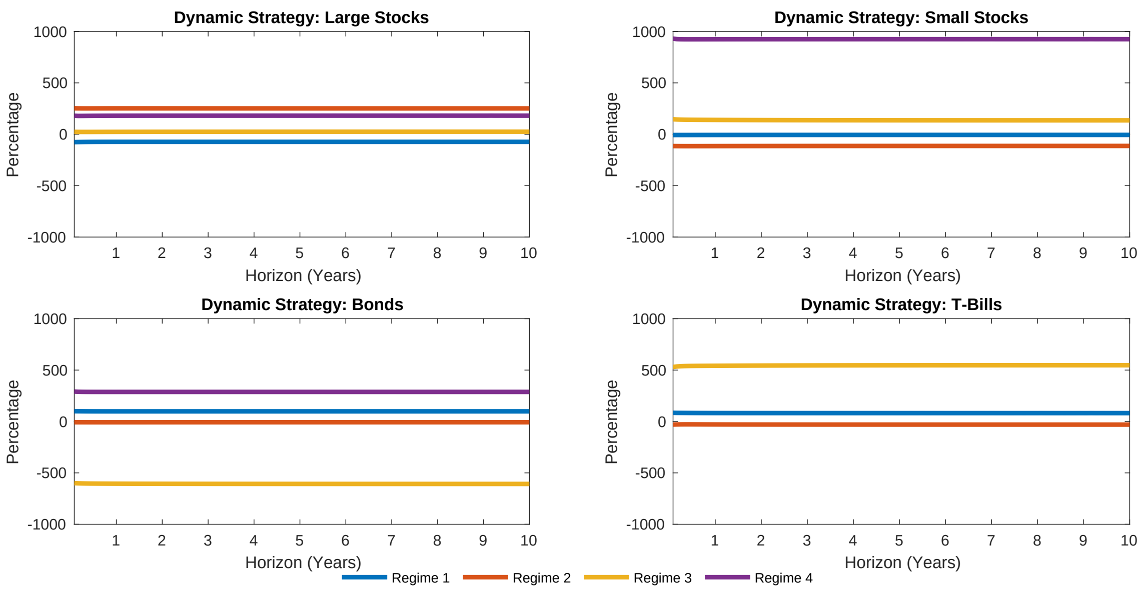


Figure 3: Four-Regime Model With Four Assets Without Predictability: Dynamic strategies for different horizons when the current regime is unknown. As the current regime is unknown, we plot the strategies when the current probabilities are either the ergodic ones or equal probabilities for each state. Short-selling is allowed. The horizon is measured in years. Investor's relative risk aversion is set $\gamma = 5$.

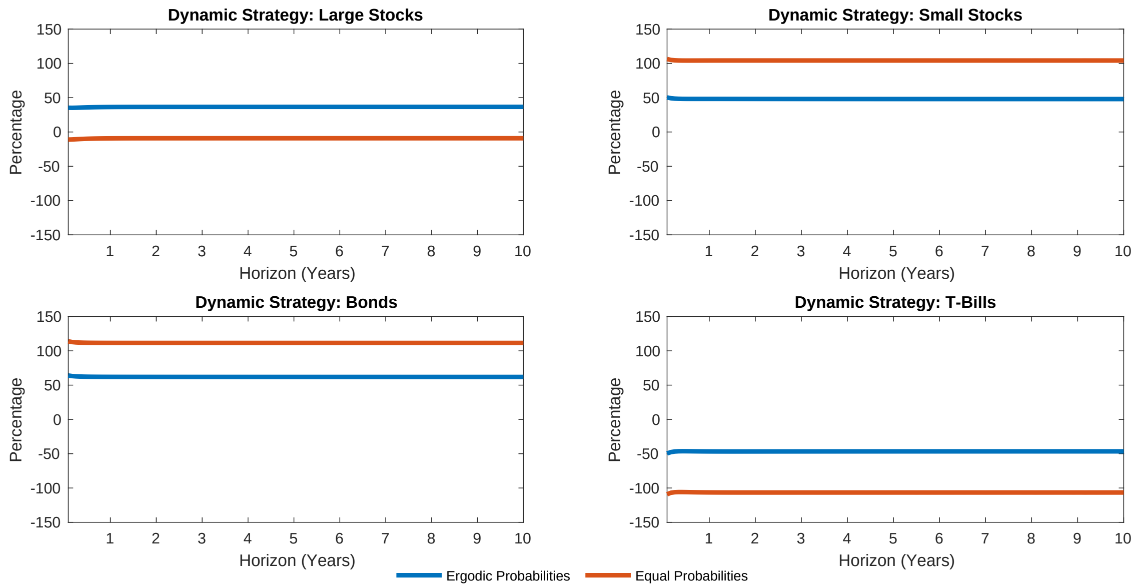
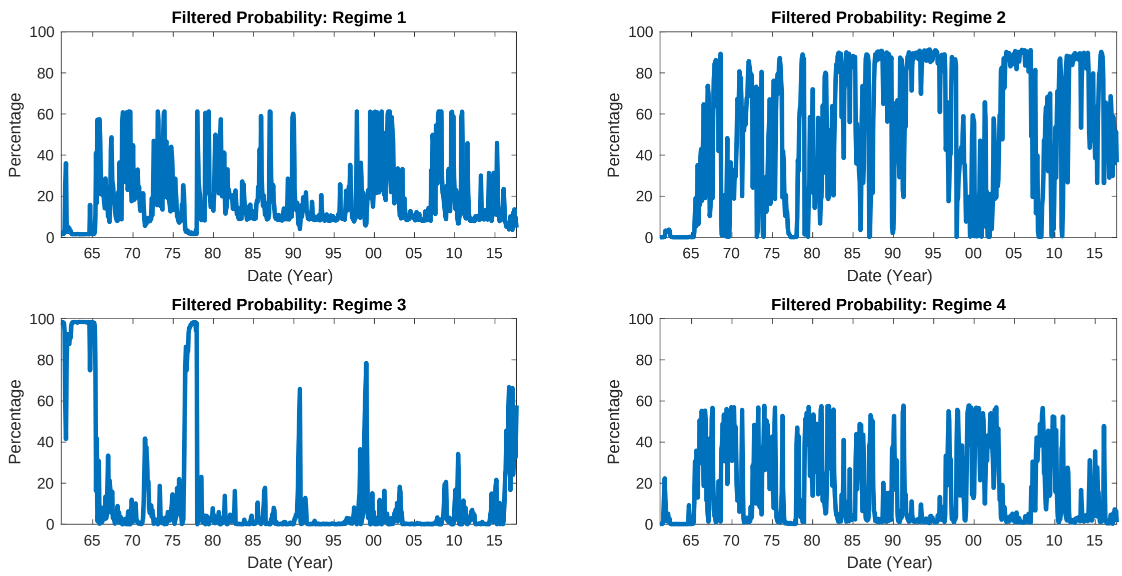


Figure 4: Four-Regime Model with Four Assets With Predictability: Filtered probabilities of being in each of the states. For every point in time, these probabilities are estimated using only the information available before time t. We begin our filter in January 1952. The data sample is from July 1962 to July 2018.



9 Tables

Table 1: Four-Regime Model with Four Assets With Predictability: Volatility matrix and ergodic probabilities. The estimation procedure is explained in the paper Section 5.1. Data are from January, 1953 to July, 2018. The table also reports the ergodic (unconditional) probabilities and the average duration of each regime.

	Z_1	Z_2	Z_3	Z_p	Ergodic Prob.	Average Duration
Regime 1	-5.34%	-1.99%	0.33%	-0.03%	15.34%	3 months
Regime 2	4.34%	-0.47%	-0.59%	0.03%	38.23%	14 months
Regime 3	0.92%	1.03%	-0.21%	0.02%	32.40%	72 months
Regime 4	0.08%	1.43%	0.47%	-0.03%	14.02%	2 months

Table 2: Single- and Four-Regime Model with Five Assets: Parameter estimates. We report in Panel A the parameters and the optimal weights for the single state model (for $\gamma = 5$). The optimal weights for the single-state model sum up to 100% by short selling the riskless asset. Panel B presents the monthly parameter estimates for the four-state model from January 1953 to July 2018. Both panels do not allow for predictability. The volatilities are reported on the diagonals of the correlation/volatilities matrices. The average Treasury yield over the full period was estimated at 0.36% a month.

Five-Assets Model				
Panel A: Single State Model	Growth	Neutral	Value	LT Bonds
Mean Returns	0.92%	1.05%	1.24%	0.49%
Correlation Matrix				
Growth	4.62%			
Neutral	82.93%	4.09%		
Value	76.10%	86.85%	5.02%	
LT Bonds	0.45%	0.38%	0.01%	2.11%
Optimal Weights	-36.07%	52.21%	59.45%	62.60%
Panel B: Four State Model	Growth	Neutral	Value	LT Bonds
Mean Returns				
Regime 1 (Crash)	-0.22%	0.09%	0.54%	0.54%
Regime 2 (Slow Growth)	1.01%	1.11%	1.08%	0.33%
Regime 3 (Bull)	1.21%	1.36%	1.60%	0.09%
Regime 4 (Recovery)	1.63%	1.64%	1.85%	0.96%
Correlation Matrices				
Regime 1 (Crash) - Growth	6.56%			
Neutral	79.36%	6.01%		
Value	77.06%	87.47%	7.52%	
LT Bonds	-0.82%	-0.83%	-13.70%	2.56%
Regime 2 (Slow Growth) - Growth	3.06%			
Neutral	80.59%	2.86%		
Value	71.11%	86.59%	4.16%	
LT Bonds	-0.38%	-0.48%	-29.98%	1.61%
Regime 3 (Bull) - Growth	3.93%			
Neutral	87.08%	3.19%		
Value	82.86%	86.23%	3.88%	
LT Bonds	0.37%	0.24%	4.14%	0.62%
Regime 4 (Recovery) - Growth	4.44%			
Neutral	88.90%	3.72%		
Value	78.23%	88.96%	3.62%	
LT Bonds	2.05%	2.13%	49.16%	2.77%
Transition Probabilities	Regime 1	Regime 2	Regime 3	Regime 4
Regime 1 (Crash)	83.99%	10.67%	0.00%	5.34%
Regime 2 (Slow Growth)	6.94%	90.35%	2.71%	0.00%
Regime 3 (Bull)	0.00%	4.56%	95.43%	0.00%
Regime 4 (Recovery)	5.18%	0.00%	0.00%	94.82%

Table 3: Four-Regime Model with Five Assets: Portfolio strategy with approximate shares. We present the approximate solution for five assets under the four-state model without predictability. The approximate shares are computed with the approximate analytical formulas presented in the text. Results are reported for different horizons, and portfolios are unconstrained (*i.e.*, short-selling is allowed). We consider that the next period regime can be known (the investor sets the filtered probability $\mathsf{P}(\mathsf{Y}_{t+1} = i|\mathfrak{F}_t)$ to 1) or unknown (in which case the investor uses the steady-state probabilities of the regimes).

Horizon (months)	**Growth**	**Neutral**	**Value**	**LT Bonds**	**Risk Free**
			Regime 1 (Crash)		
1	-63.7%	-55.3%	91.4%	61.5%	66.1%
6	-63.2%	-54.4%	90.5%	60.2%	67.0%
12	-63.3%	-54.8%	90.5%	59.0%	68.6%
60	-63.6%	-55.3%	90.7%	56.3%	71.9%
120	-63.7%	-55.4%	90.8%	56.1%	72.2%
			Regime 2 (Slow Growth)		
1	-32.0%	-26.8%	275.0%	20.9%	-137.2%
6	-33.0%	-30.7%	278.9%	16.9%	-132.1%
12	-33.6%	-31.5%	280.4%	14.0%	-129.3%
60	-35.0%	-31.2%	282.6%	9.1%	-125.6%
120	-35.1%	-31.1%	282.8%	8.8%	-125.5%
			Regime 3 (Bull)		
1	-140.7%	228.1%	132.2%	-1376.9%	1257.4%
6	-141.2%	223.7%	134.0%	-1374.1%	1257.6%
12	-140.7%	221.9%	133.9%	-1372.0%	1256.8%
60	-139.6%	220.4%	133.3%	-1371.1%	1257.0%
120	-139.7%	220.4%	133.3%	-1371.8%	1257.7%
			Regime 4 (Recovery)		
1	7.9%	314.3%	-108.0%	-14.4%	-99.7%
6	7.1%	310.8%	-106.7%	-14.9%	-96.3%
12	7.1%	309.9%	-106.8%	-15.6%	-94.6%
60	7.1%	309.5%	-107.3%	-18.3%	-91.0%
120	7.1%	309.5%	-107.4%	-18.6%	-90.6%
			Steady-State (Ergodic) Probabilities		
1	-42.9%	66.5%	62.4%	67.1%	-53.0%
6	-43.3%	64.4%	63.3%	65.4%	-49.7%
12	-43.4%	63.5%	63.4%	64.0%	-47.6%
60	-43.6%	62.9%	63.5%	61.0%	-43.8%
120	-43.6%	62.9%	63.5%	60.7%	-43.5%

Table 4: Four-Regime Model with Five Assets: Out-of-sample performance. We compare the certainty-equivalent of the out-of-sample returns obtained with three portfolio models: the four-regime model, the single-regime model and the equally-weighted model. The horizon is set to 10 years and we rebalance the portfolios monthly. For the four-regime and single-regime models we computed the optimal portfolios with the risk aversion parameter $\gamma = 5$. To give a broader perspective of the performance over time, we start the out-of-sample exercise at different times, namely January 1960, January 1980 and January 2000. Samples end in July 2018. Panel A reports the first four unconditional moments of the returns for each strategy as well as the autocorrelation of order 1 and the turnover (monthly average of the traded shares divided by the portfolio size. Panel B reports the differences in the annualized certainty-equivalent return (ΔCER) between the four-regime and the two competing models with 95% bias-corrected and accelerated bootstrap confidence intervals.

Panel A: Descriptive Statistics of Out-of-Sample Returns

Portfolios	Mean (%)	Standard Deviation (%)	Skewness	Kurtosis	Auto-correlation (%)	Turnover (%)
Since 1960						
Four-Regime	1.54	6.03	0.05	2.95	10.50	17.90
Single-Regime	1.14	4.10	-0.06	1.73	9.09	1.33
Equally-weighted	0.80	2.70	-0.31	1.57	6.22	1.30
Since 1980						
Four-Regime	1.53	5.53	-0.37	2.99	10.06	15.68
Single-Regime	1.23	4.03	-0.34	1.54	10.65	1.36
Equally-weighted	0.88	2.68	-0.53	1.94	12.67	1.33
Since 2000						
Four-Regime	1.14	4.00	-0.35	0.32	7.35	19.31
Single-Regime	0.99	4.04	-0.51	2.31	7.85	1.37
Equally-weighted	0.57	2.55	-0.51	1.37	11.00	1.32

Panel B: Δ CER	Since 1960	Since 1980	Since 2000
Four-Regime *vs.* Single-Regime	0.017 [-0.043 – 0.074]	0.016 [-0.024– 0.075]	0.015 [-0.013 – 0.068]
Four-Regime *vs.* Equally-Weighted	0.040 [-0.014 – 0.116]	0.013 [-0.054 – 0.057]	0.066 [0.064 – 0.091]

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