Optimal Portfolio Strategies in the Presence of Regimes in Asset Returns*

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Abstract

This paper analyzes optimal portfolio and consumption strategies in a regime switching economy with unobservable states and predictability of risky asset returns. We develop approximate analytical solutions to the unconstrained dynamic problem. The approximation is shown to be accurate in a four-regime setting with an allocation to large stocks, small stocks, bonds and Treasury bills. While the portfolio policy depends strongly on the current state of the economy, the consumption-to-wealth ratio is roughly state-independent. Predictability changes considerably the optimal portfolios. Hedging demands are negligible with regimes and no predictability, but can be important with predictability. On the other hand, the consumption-to-wealth ratio is not very impacted by the predictor. An out-of-sample exercise shows the economic importance of regimes.

JEL classification: G11, C02

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*This paper includes an online appendix in which we provide detailed derivations and extensions at `http://ssrn.com/abstract=BLABLABLA`.

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1 Introduction

To allocate their wealth between several asset classes over time, investors have to determine how asset returns evolve dynamically. They could assume simple processes with constant coefficients but they will be at odds with the numerous changes that affect their means, volatilities and correlations. In the recent global financial crisis of 2008-2009, stock returns exhibited drastic changes in their time-series characteristics. Since the seminal paper of Hamilton (1989), Markov switching models have been widely used to capture these sudden changes in financial markets (see a survey by Ang and Timmermann (2012)). Investors face a challenge since these regimes are not observable with certainty and they need to infer their probabilities of occurrence and their duration based on the observed returns and possibly some predictive variables. Moreover, they need to use involved numerical techniques to solve their optimal dynamic portfolio allocation problem. In this paper we provide approximate solutions to these complex problems that prove to be accurate enough to offer practical solutions to portfolio managers.

We assume that investors solve a general finite horizon portfolio allocation problem with consumption, in which we disentangle risk aversion from the elasticity of intertemporal substitution using stochastic differential utility. We allow for the presence of a predictor variable p_t in a multi-regime economy. The regimes are hidden, and investors have to estimate the probability π_t of each state at each point in time. Our general solution admits a wealth-separable solution of the form:

$$V\left(W_t, p_t, \pi_t, \tau\right) = H\left(p_t, \pi_t, \tau\right) \frac{W_t^{1-\gamma}}{1-\gamma}, \tag{1}$$

in which $\tau = T - t$ is the time left till the final horizon. We find an approximate linear expression for $H\left(p_t, \pi_t, \tau\right)$ in terms of the state variables p_t, π_t and their squares. The coefficients multiplying the state variables are horizon-dependent and solve a system of ordinary differential equations.

The optimal portfolio strategy contains the usual myopic allocation and two hedging demands related to the predictor and to the regime probabilities, all inversely proportional to the coefficient of risk aversion. The three components depend also implicitly on the elasticity of intertemporal substitution (EIS). Optimal consumption is affected by the state variables through the function $H\left(p_t, \pi_t, \tau\right)$ and is directly impacted by the EIS.

To assess the accuracy of our solution method, we use a model similar to Guidolin and Timmermann (2007), with four regimes in the returns of large and small stocks and bonds.

We compare our approximate solution to the optimal numerical solution based on Monte Carlo methods proposed in Guidolin and Timmermann (2007). We show that the approximate portfolio shares are very close to the optimal ones when regimes are assumed to be known and of similar sign and magnitude when the investor needs to estimate the probabilities of being in each of the states. We also measure the wealth equivalent utility loss created by using the approximate solution instead of the optimal solution obtained by simulation and conclude that it is negligible.

To evaluate the economic importance of regimes we conduct an out-of-sample exercise to compare the returns of a portfolio allocation to large stocks, small stocks, bonds and a risk-free asset for a four-regime model, a single-regime model and a 1/N strategy. We assume an investment horizon of 10 years with a monthly rebalancing of the portfolio. The four-regime portfolio clearly dominates the single-regime strategy and the equal-weights strategy in terms of average returns and is robust over several sub-samples. While its volatility is higher than the single-regime strategy, its Sharpe ratio is slightly better.

When stock returns are predictable by the dividend yield, the dynamic strategies remain stable when the regimes are known. However, hedging demands with respect to the predictor are important, especially for stocks because of a positive correlation with dividend yields. We also show that hedging demands increase with high values for the predictor. This contrasts with negligible hedging demands when the regime is unobservable and no predictor is used. Accounting for intermediate consumption in the investor's problem, we conclude that the consumption-to-wealth ratio does not vary much with the predictor and with the regimes.

Several papers have studied dynamic optimal portfolio problems under regime-switching processes in discrete time when regimes are unobservable. Ang and Bekaert (2002) solved numerically a two-regime portfolio problem with international assets[1], while Guidolin and Timmermann (2007) use Monte Carlo techniques to solve an allocation between large and small stocks and bonds in a four-state regime switching model. In continuous time, Liu (2011) examines a consumption-portfolio problem in which the expected return of a single risky asset follows a hidden Markov chain, under ambiguity. The investor's optimal choices are not in closed-form, but characterized in terms of Malliavin derivatives and stochastic integrals.

Several papers analyze the problem of dynamic portfolio allocation with fully observable regimes. Yin and Zhou (2003) find an explicit solution when an investor minimizes the variance

[1] See also a more recent paper by Solnik and Watevai (2016) that decomposes returns into frequent-but-small diffusion and infrequent-but-large jumps for many countries.

of a given fixed expected terminal wealth and does not consume. Sotomayor and Cadenillas (2009) study the consumption-portfolio problem of a power utility investor who maximizes the expected total discounted utility of consumption and find optimal exact portfolio and consumption policies. All market parameters (interest rate, stock drift and volatility) are constant within a regime. They work under infinite horizon and hence are not able to capture and analyze horizon effects[2]. Lim and Watevai (2012) consider the problem of optimal asset allocation for a regime switching model with jumps. They find semi-explicit expressions for the optimal policy in terms of the solution of a system of ordinary differential equations with CRRA utility. Canakoglu and Ozekici (2012) also find explicit solutions for optimal portfolio policy for HARA utilities in terms of solutions of ordinary differential equations[3]. Xing (2017) and Kraft et al. (2017) solve optimal portfolio and consumption problems with Epstein-Zin recursive utility without regime switching.

Several other papers are related to our work. Graflund and Nilsson (2003) study the relevance of intertemporal hedging and regimes under a dynamic portfolio problem with no consumption in which the investor maximizes power utility from terminal wealth. The discrete-time setting includes one risky asset and a riskless bond, in which the investor rebalances the portfolio monthly. The problem is again solved numerically, conditional on the current observable regime, with the number of regimes being determined with a Monte Carlo likelihood ratio test. They conclude that ignoring the regimes for long-horizon investors is costly, while intertemporal hedging is present in some regimes but not in others.

The importance of regime shifts for modeling asset returns has been examined by a number of researchers. For example, Ang and Bekaert (2004) show the importance of regime switching models in tactical asset allocation. Similarly, Tu (2010) concludes that certainty-equivalent losses associated with ignoring regime switching in portfolio decisions are generally above 2% per year. Guidolin and Hyde (2012) show that vector autoregressive models cannot capture regime shifts in asset returns. They use a three-regime model in which the investor has power utility. All these papers are set in discrete-time and solve the problem numerically, via Monte Carlo methods.

[2] Under unobservable regimes, Honda (2003) finds a closed-form analytical solution to the consumption and portfolio problem only for the case where the investor has a constant relative risk aversion equal to 0.5. For all other values, he uses the martingale approach to numerically solve for the optimal policies with Monte Carlo simulation.

[3] See also Celikyurt and Ozekici (2007) who study mean-variance problems with regimes and Lopez and Serrano (2015) who find closed-form expressions of the optimal value function in a pure-jump model for agents with log-utility and fractional power utility in the case of two-state Markov chains.

Guidolin and Timmermann (2006, 2008a) use a discrete-time regime switching model for the asset returns to solve the portfolio problem (with no consumption), in which the investor has utility over moments of the terminal wealth distribution. The current regime is observable and the approximate optimal portfolio weights are found as the roots of a system of polynomial equations (first-order conditions). Under unobservable regimes, Guidolin and Timmermann (2005, 2007) specify a four-state regime switching model in discrete-time with a richer risky asset menu (one large-stock index, one small-stock index and a ten-year bond). Their optimal choices, under power time-additive utility, are found numerically, with Monte Carlo techniques. Using the same techniques, Guidolin and Timmermann (2008b) consider an allocation over size and value portfolios.

The rest of the paper is structured as follows. Sections 2 and 3 describe the economy and the investor's problems respectively. The solution and its approximation are explained in Section 4. Section 5 sets up a four-regime model for small and large stocks, long-term government bonds and Treasuries, computing optimal portfolios for various scenarios. The accuracy of the approximate solution is assessed in Section 5.3 for this four-regime model. An out-of-sample exercise is conducted in Section 5.4.2. Section 5.5 introduces a predictor and shows how it affects optimal strategies in the same four-regime economy. The case with intermediate consumption is discussed in Section 5.6 while Section 6 concludes. An appendix provides all the necessary details about the approximate Bellman equations and the Monte Carlo simulation procedures to assess the accuracy of the approximate solution.

2 The Economy

We consider an economy with a short-term riskless asset and n different risky assets where the uncertainty affecting the evolution of their returns is governed by a continuous Markov chain Y_t and $n+1$ independent Brownian processes (one for each risky asset, $Z_{1,t}$, $Z_{2,t}$, ..., $Z_{n,t}$ and one for a single predictor $Z_{p,t}$). The short-term riskless asset has price M_t at time t and follows the deterministic process $\frac{dM_t}{M_t} = r dt$[4]. We first describe the Markov process Y_t and then the dynamics of the risky assets and the predictor variable.

[4] The assumption of a constant instantaneous interest rate r is often made for simplification in portfolio models with regime shifts; see, for example, Guidolin and Timmermann (2007), Liu (2011) or Honda (2003). However, Ang and Bekaert (2002) and Guidolin and Timmermann (2008b) consider the case of regime-switching short-term rates that predict future investment opportunities.

2.1 The regime-switching state variable

The regime-switching state variable Y_t is an independent continuous-time Markov chain, right-continuous and admitting only values in $R = \{1, 2, ..., m\}$, where R represents the finite set of m possible regimes in the economy. The regime-switching process Y_t, starting at any given time t_0 on a given state, remains in this state for an exponentially distributed length of time, and then jumps to another state[5]. More precisely, considering the current state is i, the probability of jumping to another state j over the next Δt time period is $P_{ij,\Delta t} = \frac{\lambda_{ij}}{\sum\limits_{k\neq i}\lambda_{ik}}\left(1 - e^{-\sum\limits_{k\neq i}\lambda_{ik}\Delta t}\right)$[6], with $j \neq i \in R$ and $\lambda_{ij} \geq 0$. We define $\lambda_{ii} = -\sum\limits_{k\neq i}\lambda_{ik} \leq 0$ such that $P_{ij,\Delta t} = \frac{\lambda_{ij}}{-\lambda_{ii}}\left(1 - e^{\lambda_{ii}\Delta t}\right)$. The probability of *staying* in the same regime i over the next Δt time period is therefore given by $P_{ii,\Delta t} = e^{\lambda_{ii}\Delta t}$.

The parameters λ_{ij} $(j \neq i \in R)$ are assumed to be constant and represent the density of transition probabilities from regime i to regime j – (as $Lim_{\Delta t \to 0}\frac{P_{ij,\Delta t}}{\Delta t} = \lambda_{ij}$). The closer λ_{ii} is to zero, the more persistent regime i is. When $\lambda_{ii} = 0$, once the economy jumps to state i, it will remain there forever. We also adopt the standard assumption that the inter-regime times are independent, and are also independent of Brownian motions governing the risky assets and the predictor.

2.2 The risky assets and the predictor

The dynamics of the n risky assets is given by:

$$\underbrace{\begin{bmatrix} \frac{dS_{1,t}}{S_{1,t}} \\ \frac{dS_{2,t}}{S_{2,t}} \\ ... \\ \frac{dS_{n,t}}{S_{n,t}} \end{bmatrix}}_{\frac{\mathbf{dS_t}}{\mathbf{S_t}}} = \underbrace{\begin{bmatrix} \mu_{1,t} \\ \mu_{2,t} \\ ... \\ \mu_{n,t} \end{bmatrix}}_{\mu_{\mathbf{s,t}}} dt + \underbrace{\begin{bmatrix} \sigma_{11} & 0 & ... & 0 \\ \sigma_{21} & \sigma_{22} & ... & 0 \\ ... & ... & ... & 0 \\ \sigma_{n1} & \sigma_{n2} & ... & \sigma_{nn} \end{bmatrix}}_{\sigma_{\mathbf{s}}} \underbrace{\begin{bmatrix} dZ_{1,t} \\ dZ_{2,t} \\ ... \\ dZ_{n,t} \end{bmatrix}}_{\mathbf{dZ_t}}, \tag{2}$$

where $Z_{1,t}$, $Z_{2,t}$, ..., $Z_{n,t}$ are standard and independent Brownian motion processes, $\mu_{s,t}$ is a $n \times 1$ column vector of instantaneous expected risk-premia and σ_s a $n \times n$ lower triangular constant volatility matrix.

We also assume that there exists a predictor variable p_t of the risk premia of the risky assets

[5] An important property of the exponential distribution is that it is memoryless, in accordance to the Markov property. This *memorylessness* can be stated as follows: $P(T > s + t | T > s) = P(T > t)$ for all $s, t \geq 0$, where T is the length-of-time process in a state.

[6] To be concise, $P_{ij,\Delta t} = P(Y_{\Delta t} = j | Y_t = i), \forall t \in [0, \Delta t]$.

that follows a diffusion process:

$$dp_t = \mu_{p,t} dt + \sigma_p \underbrace{\begin{bmatrix} dZ_{1,t} & dZ_{2,t} & \dots & dZ_{n,t} & dZ_{p,t} \end{bmatrix}^T}_{dZ_t^*}, \tag{3}$$

with $\sigma_p = \begin{bmatrix} \sigma_{p1} & \sigma_{p2} & \dots & \sigma_{pn} & \sigma_{pp} \end{bmatrix}$ a constant row vector with $n+1$ elements, and $Z_{p,t}$ is another standard Brownian motion, independent from the previously defined Brownian motions. We concatenate the risky assets and predictor volatility matrices in a $(n+1) \times (n+1)$ lower triangular volatility matrix σ: $\sigma = \begin{bmatrix} \sigma_s \\ \sigma_p \end{bmatrix}$ and $\Sigma = \sigma\sigma^T$.

The drifts of the risky assets and the predictor are regime-dependent and therefore a function of the unobservable state variable Y_t.

3 The Investor's Problems

Investors do not observe the drifts of the risky assets and the predictor nor the underlying Brownian motions but they can learn about them by observing the returns of the risky assets and the predictor. Investors have therefore a filtering problem to solve to infer the probabilities of being in each regime as well as the probabilities of shifting between regimes. Given these probabilities, they can solve an optimal consumption and portfolio allocation problem.

3.1 The filtering problem

We rely on the model with incomplete information about growth regimes introduced in David (1997), Veronesi (2000), David and Veronesi (2013), and extended recently by Berrada, Detemple and Rindisbacher (2018). Let us define the drift vector $\mu_t = (\mu_{s,t}, \mu_{p,t})$ that collects the drifts of the n risky assets and the drift of the predictor. Given the Markov economy defined in the last section, this drift vector will follow a continuous-time Markov chain with m regimes. In each regime, the predictor will have a different drift that we collect in a vector $\mathbf{D_p}$:

$$\mathbf{D_p} = \begin{bmatrix} \mu_{p,1} & \mu_{p,2} & \dots & \mu_{p,m} \end{bmatrix}. \tag{4}$$

The risky asset drifts are linear with the predictor (thus time-varying even if the regime does not change) and also regime-dependent. We collect these drifts in a drift matrix $\mathbf{D}_{s,t}$ of dimension nxm, which is time-varying and dependent on the predictor's value p_t[7]:

$$\mathbf{D_{s,t}} = \underbrace{\begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,m} \\ a_{2,1} & a_{2,2} & \dots & a_{2,m} \\ \dots & \dots & \dots & \dots \\ a_{n,1} & a_{n,2} & \dots & a_{n,m} \end{bmatrix}}_{\mathbb{A}} + \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_n \end{bmatrix}}_{\mathbb{B}} \underbrace{\begin{bmatrix} 1 & 1 & \dots & 1 \end{bmatrix}}_{m \text{ columns}} p_t. \tag{5}$$

Denote μ_i the drift of μ_t in regime i. Based on their information set $\mathfrak{F}_t$, investors form subjective probabilities of regime i as: $\pi_{it} = \text{prob}(\mu_t = \mu_i|\mathfrak{F}_t)$. From Wonham (1964) the probabilities $\pi_t = (\pi_{1t}, \dots, \pi_{mt})^T$ follow the process[8]:

$$d\pi_t = \Lambda' \pi_t dt + \Sigma_\pi(\pi_t) d\tilde{Z}_t, \tag{6}$$

where $\Lambda = \{\lambda_{i,j}, i = 1, \dots, m, j = 1, \dots, m\}$ and $\Sigma_\pi(\pi_t)$ the $(m \times (n+1)$ matrix with i^{th} row given by:

$$\sigma_i(\pi_t) = \pi_{it}[\mu_i - \bar{\mu}(\pi_t)]' \Sigma^{-1}, \tag{7}$$

with $\bar{\mu}(\pi_t) = \sum_{i=1}^n \pi_{it}\mu_i$. The $(n+1) \times 1$ vector $d\tilde{Z}_t$ is defined as:

$$d\tilde{Z}_t = \Sigma^{-1}(\mu_t - \bar{\mu}(\pi_t))dt + dZ_t^*. \tag{8}$$

Suppose now that we want to make the volatilities of the risky assets dependent on the Markov chain: $\sigma_{s,t} = \sigma_s \otimes Y_t$, where $\otimes$ denotes the Kronecker product, that is m volatility matrices, one per regime. In this case, in continuous time, the dependence of the volatility on the same Markov chain as the drift makes the Markov chain observable, as shown by Krishnamurthy, Leoff and Sass (2018)[9].

To get a model which is both consistent and has good econometric properties, Haussmann and Sass (2004) introduce stochastic volatility in a continuous-time HMM (Hidden Markov Model). They propose a regime-switching model where the volatility depends on an observable diffusion. They show that despite the non-constant volatility the chain is still hidden and they derive its finite-dimensional filtering equation. Krishnamurthy, Leoff and Sass (2018) construct,

[7] We assume that the b are not regime-dependent for simplicity since we expect that the intercepts (the parameters a) will vary more with the regimes than the slopes. It seems to be the case in the paper by Guidolin and Timmermann (2008b).

[8] See David and Veronesi (2013).

[9] See their proposition 2.1.

for a given filtration, the volatility process adapted to this filtration that best approximates the MSM-returns (Markov Switching Models) in the mean squared sense. They propose a so-called filter-based volatility model where: $\sigma_{s,t} = \sigma_s \otimes \mathbb{E}[Y_t|\mathfrak{F}_t^{FB}]$, where $\mathfrak{F}_t^{FB}$ is the filtration generated by the observed returns. Therefore the volatilities are adapted to this observation filtration.

We can therefore rewrite the filtered processes for the risky assets and the predictor conditional to the observed returns and predictor filtration as:

$$\frac{\mathbf{dS_t}}{\mathbf{S_t}} = \mathbf{D_{s,t}}\pi_\mathbf{t}\,dt + (\mathbf{V}\pi_\mathbf{t})\,\mathbf{d\tilde{Z}_t}, \text{ and} \tag{9}$$

$$dp_t = \mathbf{D_p}\pi_t\,dt + \sigma_p\,d\tilde{Z}_t^*. \tag{10}$$

The matrix $\mathbf{V}$ is a 1 x m row vector of matrices:

$$\mathbf{V} = \begin{bmatrix} \sigma_{\mathbf{s,1}} & \sigma_{\mathbf{s,2}} & \dots & \sigma_{\mathbf{s,m}} \end{bmatrix}, \tag{11}$$

where[10]:

$$\sigma_{\mathbf{s,i}} = \begin{bmatrix} \sigma_{11,i} & 0 & \dots & 0 \\ \sigma_{21,i} & \sigma_{22,i} & \dots & 0 \\ \dots & \dots & \dots & 0 \\ \sigma_{n1,i} & \sigma_{n2,i} & \dots & \sigma_{nn,i} \end{bmatrix}. \tag{12}$$

3.2 The wealth process and preferences

In this setting the investor consumes and invests in assets. Wealth is the value of the portfolio of assets held by the investor. Let W_t denote wealth at time t. If α_t is the $1 \times n$ vector of proportions of wealth invested in the risky assets (hence $1 - \alpha_\mathbf{t}\mathbf{1}$ is the proportion invested in the riskless asset) and C_t the amount withdrawn from the portfolio for consumption then wealth evolves according to:

$$\begin{aligned} dW_t &= (1 - \alpha_\mathbf{t}\mathbf{1})\,W_t\frac{dM_t}{M_t} + W_t\alpha_\mathbf{t}\frac{\mathbf{dS_t}}{\mathbf{S_t}} - C_t dt \\ &= (1 - \alpha_\mathbf{t}\mathbf{1})\,W_t r dt + W_t\alpha_\mathbf{t}\left[\mathbf{D_{s,t}}\pi_\mathbf{t} dt + (\mathbf{V}\pi_\mathbf{t})\,\mathbf{dZ_t}\right] - C_t dt \\ &= W_t r dt + W_t\alpha_\mathbf{t}\left[(\mathbf{D_{s,t}}\pi_\mathbf{t} - r\mathbf{1})\,dt + (\mathbf{V}\pi_\mathbf{t})\,\mathbf{dZ_t}\right] - C_t dt, \end{aligned} \tag{13}$$

where $\mathbf{1}$ represents a column vector of n ones and $\alpha_\mathbf{t} = [\alpha_{1,t}\ \alpha_{2,t}\ \dots\ \alpha_{n,t}]$.

[10] The matrices $\sigma_{\mathbf{s,i}}$ are constant and specific to the regime ($i \in R$) and are defined as lower triangular without loss of generality.

Preferences: Given a time-T finite horizon, the investor's preferences over consumption and terminal wealth (bequest) are represented by a continuous-time recursive utility function, also known as stochastic differential utility function, introduced by Duffie and Epstein (1992):

$$J_t = E_t \left[\int_{u=t}^{T} f(C_u, J_u)\, du + \frac{W_T^{1-\gamma}}{1-\gamma} \right]. \tag{14}$$

In equation (14), $f(C, J)$ is a normalized aggregator of current consumption and the continuation utility that takes the following form:

$$f(C, J) = \frac{\beta}{1-\frac{1}{\psi}} (1-\gamma) J \left\{ \left[\frac{C}{[(1-\gamma) J]^{\frac{1}{1-\gamma}}} \right]^{1-\frac{1}{\psi}} - 1 \right\}, \tag{15a}$$

where $\beta > 0$ is a rate of time preference, γ a parameter that controls the investor's risk attitudes over the states of nature, while ψ is related to the investor's consumption choices over time. We denote γ as the (relative) risk aversion and ψ as the elasticity of intertemporal substitution.

There are two special cases of the normalized aggregator given by equation (15a): $\psi = \frac{1}{\gamma}$ and $\psi = 1$. The first case is the standard, time-additive power utility function[11], where log-utility optimal choices are obtained as the limit of the respective optimal choices when $\gamma \to 1$. In the second case, interpreted as the limit when $\psi \to 1$, the aggregator takes the following limit form:

$$\psi = 1 \qquad \to \qquad f(C, J) = \beta (1-\gamma) J \ln \left\{ \frac{C}{[(1-\gamma) J]^{\frac{1}{1-\gamma}}} \right\}. \tag{15b}$$

When $\psi = 1$ the investor consumes myopically and ignores investment opportunities. When $\psi < 1$, income effects prevail and better opportunities increase consumption. When $\psi > 1$, substitution effects dominate as the investor is willing to postpone consumption and take advantage of improved investment opportunities.

4 Solving the Problem

4.1 Consumption and portfolio policies

We want to choose the consumption C_t and portfolio weights α_t that maximize the value function $V(W_t, p_t, \pi_t, t) = \sup_{\{C_t, \alpha_t\}} [J(W_t, p_t, \pi_t, t)]$. Given (10) and (13), under Ito's rule, we

[11] Note that the resulting formulation may not take the standard power utility form, but will imply the same underlying preferences, and hence the same consumption and asset allocation choices.

can write the Bellman equation that represents the recursive problem:

$$0 = \sup_{\{C_t,\alpha_t\}} \left[\begin{array}{c} f(C_t, J_t) + \frac{\partial J_t}{\partial t} + J_w \left[W_t r + W_t \alpha_\mathbf{t} \left(\mathbf{D_{s,t}} \pi_\mathbf{t} - r\mathbf{1}\right) - C_t\right] + \\ + \frac{1}{2} J_{ww} W_t^2 \alpha_\mathbf{t} \left(\mathbf{V}\pi_\mathbf{t}\right) \left(\mathbf{V}\pi_\mathbf{t}\right)^\mathbf{T} \alpha_\mathbf{t}^\mathbf{T} + J_p \mathbf{D_p} \pi_\mathbf{t} + \frac{1}{2} J_{pp} \sigma_\mathbf{p} \sigma_\mathbf{p}^\mathbf{T} + \\ + \sum_{i=1}^{m} J_{\pi_i} \sum_{j=1}^{m} \lambda_{ji} \pi_{j,t} + \frac{1}{2} \sum_{i=1}^{m} J_{\pi_i \pi_i} \sigma_{\mathbf{i},\pi} \sigma_{\mathbf{i},\pi}^\mathbf{T} + J_{wp} W_t \alpha_\mathbf{t} \left(\mathbf{V}\pi_\mathbf{t}\right) \overline{\sigma}_\mathbf{p}^\mathbf{T} + \\ + \sum_{i=1}^{m} J_{w\pi_i} W_t \alpha_\mathbf{t} \left(\mathbf{V}\pi_\mathbf{t}\right) \overline{\sigma}_{\mathbf{i},\pi}^\mathbf{T} + \sum_{i=1}^{m} J_{p\pi_i} \sigma_\mathbf{p} \sigma_{\mathbf{i},\pi}^\mathbf{T} + \sum_{i<j} J_{\pi_i \pi_j} \sigma_{\mathbf{i},\pi} \sigma_{\mathbf{j},\pi}^\mathbf{T} \end{array} \right], \qquad (16)$$

where $\overline{\sigma}_\mathbf{p}^\mathbf{T}$ and $\overline{\sigma}_{\mathbf{i},\pi}^\mathbf{T}$ represent the transposed vectors of respectively $\sigma_\mathbf{p}$ and $\sigma_{\mathbf{i},\pi}$ ($i \in R$) without their last element[12]. The subscripts denote partial derivatives (except t, that refers to the value at time t). Thus, the first-order condition for consumption in the recursive problem is given by:

$$V_w = \frac{\partial f(C_t, V_t)}{\partial C_t} = \beta (1-\gamma)^{\frac{\frac{1}{\psi}-\gamma}{1-\gamma}} V_t^{\frac{\frac{1}{\psi}-\gamma}{1-\gamma}} C_t^{-\frac{1}{\psi}}, \qquad (17a)$$

from which we solve for the optimal consumption strategy:

$$C_t = \beta^\psi V_w^{-\psi} \left[(1-\gamma) V_t\right]^{\frac{1-\psi\gamma}{1-\gamma}}, \qquad (17b)$$

which simplifies to $C_t = \beta V_w^{-1} V_t (1-\gamma)$ when $\psi = 1$. Inserting equation (17b) into (15a) and (15b) we can write the recursive aggregator in terms of the sole value function:

$$f(V_t) = \frac{1}{1-\frac{1}{\psi}} \left\{ \beta^\psi V_w^{1-\psi} \left[(1-\gamma) V_t\right]^{\frac{1-\psi\gamma}{1-\gamma}} - \beta (1-\gamma) V_t \right\} \qquad \psi \neq 1, \qquad (18a)$$

$$f(V_t) = \beta (1-\gamma) V_t \ln \left\{ \beta V_w^{-1} \left[V_t (1-\gamma)\right]^{\frac{-\gamma}{1-\gamma}} \right\} \qquad \psi = 1. \qquad (18b)$$

The first-order condition for the portfolio weights gives the following solution:

$$\begin{aligned} \alpha_\mathbf{t} &= \frac{V_w}{-V_{ww} W_t} \left(\mathbf{D_{s,t}} \pi_\mathbf{t} - r\mathbf{1}\right)^\mathbf{T} \left[\left(\mathbf{V}\pi_\mathbf{t}\right) \left(\mathbf{V}\pi_\mathbf{t}\right)^\mathbf{T}\right]^{-\mathbf{1}} + \\ &\quad + \frac{V_{wp}}{-V_{ww} W_t} \overline{\sigma}_\mathbf{p} \left(\mathbf{V}\pi_\mathbf{t}\right)^{-\mathbf{1}} + \sum_{i=1}^{m} \frac{V_{w\pi_i}}{-V_{ww} W_t} \overline{\sigma}_{\mathbf{i},\pi} \left(\mathbf{V}\pi_\mathbf{t}\right)^{-\mathbf{1}}. \end{aligned} \qquad (19)$$

The first term identifies the myopic demand while the second and third components capture the hedging demands that will be held by the investor to hedge against future undesirable movements respectively in the predictor and in the state probabilities.

[12] These last terms will not play a role in the solution since the corresponding Brownian motion has no correlation with the wealth dynamics.

To obtain the final Bellman equation for the problem under recursive utility, we substitute the optimal consumption and portfolio policies just derived into equation (16):

$$\begin{aligned}
0 \;=\; & f(V_t) + \frac{\partial V_t}{\partial t} + V_w W_t r - \beta^{\psi} V_w^{1-\psi} \left[(1-\gamma) V_t\right]^{\frac{1-\gamma\psi}{1-\gamma}} + V_p \mathbf{D_p} \pi_t + \frac{\mathbf{1}}{\mathbf{2}} V_{pp} \sigma_{\mathbf{p}} \sigma_{\mathbf{p}}^{\mathbf{T}} + \\
& + \sum_{i=1}^{m} V_{p\pi_i} \sigma_{\mathbf{p}} \sigma_{\mathbf{i},\pi}^{\mathbf{T}} + \sum_{i,j=1}^{m} V_{\pi_i} \lambda_{ji} \pi_{j,t} + \frac{1}{2} \sum_{i,j=1}^{m} \left(V_{\pi_i \pi_j} \sigma_{\mathbf{i},\pi} \sigma_{\mathbf{j},\pi}^{\mathbf{T}} - \frac{V_{w\pi_i} V_{w\pi_j}}{V_{ww}} \overline{\sigma}_{\mathbf{i},\pi} \overline{\sigma}_{\mathbf{j},\pi}^{\mathbf{T}} \right) - \\
& - \frac{1}{2} \frac{V_w^2}{V_{ww}} \left(\mathbf{D_{s,t}} \pi_t - r\mathbf{1}\right)^{\mathbf{T}} \left[(\mathbf{V}\pi_t) (\mathbf{V}\pi_t)^{\mathbf{T}} \right]^{-1} \left(\mathbf{D_{s,t}} \pi_t - r\mathbf{1}\right) - \frac{1}{2} \frac{V_{wp}^2}{V_{ww}} \overline{\sigma}_{\mathbf{p}} \overline{\sigma}_{\mathbf{p}}^{\mathbf{T}} - \\
& - \frac{V_w V_{wp}}{V_{ww}} \overline{\sigma}_{\mathbf{p}} (\mathbf{V}\pi_t)^{-1} \left(\mathbf{D_{s,t}} \pi_t - r\mathbf{1}\right) - \sum_{i=1}^{m} \frac{V_w V_{w\pi_i}}{V_{ww}} \overline{\sigma}_{\mathbf{i},\pi} (\mathbf{V}\pi_t)^{-1} \left(\mathbf{D_{s,t}} \pi_t - r\mathbf{1}\right) - \\
& - \sum_{i=1}^{m} \frac{V_{wp} V_{w\pi_i}}{V_{ww}} \overline{\sigma}_{\mathbf{p}} \overline{\sigma}_{\mathbf{i},\pi}^{\mathbf{T}}.
\end{aligned} \tag{20}$$

We also consider a problem with no consumption in which all (power) utility comes from terminal wealth[13].

Both problems, with and without consumption, will admit the wealth-separable solution:

$$V(W_t, p_t, \pi_t, \tau) = H(p_t, \pi_t, \tau) \frac{W_t^{1-\gamma}}{1-\gamma}, \tag{21}$$

with $\tau = T - t$ the time remaining until the terminal date and the terminal condition $H(p_t, \pi_t, 0) = 1$ due to the presence of bequest. The Bellman equation does not admit an analytical solution due to the presence of nonlinear terms. In the next section, we propose an approximate analytical solution based on a log-linear approximation of the function $H(p_t, \pi_t, \tau)$.

4.2 The approximate analytical solution

Our approximation is inspired by solutions to a similar consumption-portfolio problem with no regimes. In an infinite horizon setting, Campbell and Viceira (1999) and Campbell et al. (2004) provided an approximation for the discrete time case in which the log consumption ratio is approximated around its unconditional long-run mean. Campbell et al. (2004) extended this approximate solution method to a continuous-time setting. The infinite horizon is crucial since the coefficients of the log linear consumption-to-wealth ratio are constant. Campani and Garcia (2019) develop a solution when the horizon is finite. In this case, the long-run mean for consumption may not be a good proxy for short-horizon investors proxy and thus a good point at which to perform the approximation. Moreover, under finite horizon, the consumption-to-wealth

[13] See the corresponding Bellman equations in Appendix A.

ratio is time-varying even if the state variables do not vary.

They propose to approximate the consumption-to-wealth ratio around a fixed point of the state-variable space. Given the persistence of usual predictors of stock returns, they choose the long-run mean of the predictor[14]. The approximation suggested is:

$$\exp\left(\ln\frac{C_t}{W_t}\right) \approx \exp\left(\ln\frac{C_t}{W_t}\right)_{P_t=\overline{P}} + \exp\left(\ln\frac{C_t}{W_t}\right)_{P_t=\overline{P}}\left[\ln\frac{C_t}{W_t} - \left(\ln\frac{C_t}{W_t}\right)_{P_t=\overline{P}}\right]. \tag{22}$$

They guess a solution of the form: $\ln\frac{C_t}{W_t} = A_1(\tau) + A_2(\tau)P_t + \frac{A_3(\tau)}{2}P_t^2$ and obtain the following expression for the value function:

$$V(W_t, P_t, \tau) = \frac{W_t^{1-\gamma}}{1-\gamma}\exp\left\{(1-\gamma)\left[A_1(\tau) + A_2(\tau)P_t + \frac{A_3(\tau)}{2}P_t^2\right]\right\},$$

with terminal condition $V(W_t, P_t, 0) = \frac{W_T^{1-\gamma}}{1-\gamma}$.

By substituting this expression into the Bellman equation, they can solve for $A_i(\tau)$, $i = 1, 2, 3$, in a system of ordinary differential equations with boundary conditions equal to zero since the value function should converge to the power utility of wealth irrespectively of state-variable values at the final horizon.

We follow the same approach to construct our approximate solution. Our state space includes now both the predictor and the regime probabilities. Therefore, since:

$$\frac{C_t}{W_t} = \beta^{\psi}H(p_t, \pi_t, \tau)^{\frac{1-\psi}{1-\gamma}}, \tag{23}$$

we propose the following functional form for the H function:

$$H(p_t, \pi_t, \tau) = \exp\left[\begin{array}{c} A_0(\tau) + A_p(\tau)p_t + \sum_{i=1}^{m} A_i(\tau)\pi_{i,t} + B_p(\tau)p_t^2 + \\ + \sum_{i=1}^{m} B_i(\tau)\pi_{i,t}^2 + \sum_{i=1}^{m} C_{pi}(\tau)p_t\pi_{i,t} + \sum_{i<j} C_{ij}(\tau)\pi_{i,t}\pi_{j,t} \end{array}\right]. \tag{24}$$

All coefficients above are time-varying and depend on the primitive parameters of the model describing the regime-switching economy, market investment opportunities and investor's preferences. They solve a system of ordinary differential equations with boundary conditions equal to zero when $\tau = 0$ (*i.e.*, at maturity). This solution is valid for the two cases with and without

[14] However this point can be chosen by the investor according to current market conditions or expectations. This makes the approach convenient for a real-time portfolio optimization.

consumption, even though the coefficients will be different for each case. We develop the approximate Bellman equation in Appendix B by using the approximate function $H(.)$ in equation (24) for the value function and its derivatives.

Once the coefficients are obtained, the optimal portfolio strategy can be computed with the following formula:

$$\begin{aligned}
\alpha_{\mathbf{t}} &= \frac{1}{\gamma}\left(\mathbf{D}_{\mathbf{s},\mathbf{t}}\pi_{\mathbf{t}} - r\mathbf{1}\right)^{\mathbf{T}}\left[\left(\mathbf{V}\pi_{\mathbf{t}}\right)\left(\mathbf{V}\pi_{\mathbf{t}}\right)^{\mathbf{T}}\right]^{-1} + \\
&+ \frac{1}{\gamma}\left[A_p(\tau) + 2B_p(\tau)p_t + \sum_{i=1}^{m} C_{pi}(\tau)\pi_{i,t}\right]\overline{\sigma}_{\mathbf{p}}\left(\mathbf{V}\pi_{\mathbf{t}}\right)^{-\mathbf{1}} + \\
&+ \frac{1}{\gamma}\sum_{i=1}^{m}\left[A_i(\tau) + 2B_i(\tau)\pi_{i,t} + C_{pi}(\tau)p_t + \sum_{j\neq i} C_{ij}(\tau)\pi_{j,t}\right]\overline{\sigma}_{\mathbf{i},\pi}\left(\mathbf{V}\pi_{\mathbf{t}}\right)^{-\mathbf{1}}.
\end{aligned} \tag{25}$$

Note that all terms (myopic allocation and hedging demands) are proportional to the reciprocal of γ, the parameter capturing risk aversion also in the recursive framework. Our model implies that the Sharpe ratio is calculated using the risk premia and the volatility matrices as weighted averages of their respective values in the regimes, with the filtered probabilities as weights.

The optimal strategy for consumption is given by equation (23). Therefore optimal consumption is affected by the state variables through the function H and by ψ, the elasticity of intertemporal substitution. The portfolio allocation also depends on this parameter through the coefficients since all equations depend on ψ[15].

5 An Illustration: A Four-Regime Model with Large Firms, Small Firms and Long-Term Government Bonds

To illustrate the implications of the model described in the previous sections, we follow Guidolin and Timmermann (2007). This paper is a reference in two respects. It offers the most realistic application of an optimal portfolio problem in the context of Markov switching returns for the fundamentals. They select $n = 3$ risky assets (a large cap index, a small cap index and a long-term bond index) and $m = 4$ regimes and show that this specification fits best the data. For

[15] These optimal strategies nest the ones when $\psi = 1$. In this case, the consumption-to-wealth ratio remains constant and equal to β. The investor with $\psi = 1$ consumes just as the investor who sees a single regime in the economy, *i.e.*, with a constant proportional rate, even when the predictor evolves or the regime changes. As we mentioned, the income and substitution effects cancel each other in this case and the investor ignores the investment opportunities.

our purpose it offers the optimal solution to the portfolio problem based on simulation methods. Therefore it will be the best benchmark to show the accuracy of our approximation approach both in terms of complexity and usefulness of our solution method. We therefore estimate the same model and describe our results in section 5.1. Based on these estimates, we evaluate the accuracy of our approximation method in section 5.3. Finally we collect the empirical results in the remaining sections, where we compute the dynamic portfolio strategies with and without the predictor.

5.1 Estimation of the parameters of the model

To estimate the parameters, we need to write the model in discrete time. This has also a consequence on the decision process of the investor. In a discrete-time strategy, the investor makes her portfolio and consumption decisions based on current information and waits until the next period to make new decisions based on the newly acquired information. We set the rebalancing period equal to one month to follow the portfolio literature on discrete-time regime switching models (*e.g.*, Guidolin and Timmermann (2005, 2007, 2008a), Ang and Timmermann (2012), Ang and Bekaert (2002, 2004)).

For the large and small cap indices, we select the highest and lowest quintiles based on the capitalization value of all NYSE, AMEX, and NASDAQ firms[16]. The long-term bond index is represented by the CRSP portfolio of 10-year Treasury-bonds[17]. We estimate the model over the period from January 1952 to July 2018 (799 monthly observations)[18].

The parameter values are shown in Table 1. In Panel A, we show the means and correlations of the three asset classes as if there were a single regime in the economy. As expected, the highest returns are obtained for the small caps index and the highest correlation is achieved between the large caps index and the small caps one. We also report the optimal portfolio shares corresponding to this single regime with close to an equal allocation between stocks and bonds, and a slightly higher share for the large stocks than for the small stocks.

In panel B, we report the results for the four regimes. We use the same terminology to describe these regimes as Guidolin and Timmermann (2007) – crash, slow growth, bull and recovery states – even if our sample is larger. The bull-regime is good for the stocks and bad for the bonds, while the crash means are negative for the stocks and high for the bonds since there

[16] The stock indices were downloaded from Kenneth French's website.
[17] The long-term bond index was downloaded from CRSP through WRDS.
[18] Details for estimating the parameters based on Hamilton (1989) methodology are found in our online appendix.

is a flight to quality in a depressed state. The slow growth affects mainly the mean returns of the small caps and the recovery boosts all asset classes with means greater than in all other states. In terms of average probabilities (see Table 2) the slow growth regime is the most frequent (40% of the time) followed by the bull state (31%). In terms of average duration, the bull state lasts longer, way more than any other states[19]. The two extreme regimes last only a few months. From the crash regime, the economy transitions to the recovery state, while from the recovery state it transitions to the slow growth state. The correlation matrix indicates that the highest volatilities are in the worst state and the low ones in the bull state. The correlation between the large and the small stocks is close to what it was for the single regime (around 73%) except in the recovery period. The correlation between bonds and small caps is negative in all regimes.

The estimation procedure provides filtered probabilities of being in each regime at time $t+1$ for each point in time t for the whole estimation period[20]. These probabilities are conditional to the information available to investors when they make their portfolio and consumption decisions. We will use these conditional probabilities to evaluate the returns of dynamic strategies based on our approximate solution in section 5.4.2.

We still have to estimate σ_π, the matrix introduced in the filtering problem in equation (6). To estimate σ_π in a discrete time setup, we first create an $(n+1)$ x m matrix (denoted by $\mathbf{D_t}$) consisting of $\mathbf{D_{s,t}}$ in the first n rows and of $\mathbf{D_p}$ over the last row. Then we calculate the monthly time-series for the drifts $(\mathbf{D_t}\pi_\mathbf{t})$ and volatility matrix $(\mathbf{V}\pi_\mathbf{t})$ [21]. Then, following equations (3) and (9), we can estimate the discretized process:

$$\Delta\mathbf{Z}_\mathbf{t}^* = \sigma_\mathbf{t}{}^{-1}\left(\mathbf{LR_t} - \mathbf{D_t}\pi_\mathbf{t}\right), \tag{26}$$

where $\mathbf{LR_t}$ is an $(n+1)$ x 1 vector of the n risky assets observed log-returns at period t together with the predictor change in the same period (see the online appendix for more details). The matrix $\sigma_\mathbf{t}$ is the concatenation of $(\mathbf{V}\pi_\mathbf{t})$ with $\sigma_\mathbf{p}$, thus creating a square matrix of order $n+1$ (the n first elements of the last column are filled up with zeros). We use equation (6) to write the discretized process $(\Delta\pi_\mathbf{t} - \mu_{\pi,\mathbf{t}}) = \sigma_\pi\Delta\mathbf{Z}_\mathbf{t}^*$ and store the left-hand time-series in an m x T matrix denoted by $(\Delta\pi - \mu_\pi)$. We also store all increments $\Delta\mathbf{Z}_\mathbf{t}^*$ in an $(n+1)$ x T matrix denoted

[19] This is obtained by computing $1/(1-p_{ii}$, where p_{ii} is the transition probability corresponding to staying in the same state i.

[20] We report the graphs of these probabilities in the online appendix.

[21] To achieve a more robust methodology, we disregarded the first year of data. This is explained by the fact that the choice of the starting probabilities still had some effects over the filtered probabilities during this first year. These starting probabilities were revealed to be completely irrelevant after a year from the starting date.

simply by $\Delta \mathbf{Z}^*$ (where T is the time-series length). We finally obtain the desired estimation:

$$\sigma_\pi = (\Delta\pi - \mu_\pi)\,\Delta \mathbf{Z}^{*\mathbf{T}} \left(\Delta \mathbf{Z}^* \Delta \mathbf{Z}^{*\mathbf{T}}\right)^{-\mathbf{1}}. \tag{27}$$

5.2 Optimal portfolio strategies under the four-regime model

To evaluate the accuracy of our approximate solution, we will use as a benchmark the four-regime model estimated in the previous section[22]. To compute the optimal portfolio solution for a dynamic investor under a regime switching economy in a finite horizon, we will rely on the numerical solution method proposed in Guidolin and Timmermann (2007). They solve the dynamic portfolio problem of the investor numerically by using Monte Carlo methods to approximate expected utility. We describe this method in detail in Appendix C.

We report the asset shares in Table 3 under the column Optimal Shares when each of the current states is supposed to be known and when it is unknown, in which case ergodic probabilities are used. The shares are estimated for different horizons, from 1 to 120 months. Results are as expected given our estimates. In the crash state, the investor sells short heavily large stocks (less volatile than small stocks) to invest in safer assets (bonds and risk-free asset). In the slow growth regime, the investor loads on large stocks and sells short small caps and bonds, since bonds and small caps generate lower expected returns. In the bull regime, long-term bonds are hugely sold short to mainly invest in the riskless asset and in stocks (mainly small). In this regime, as in the previous one, bonds offer a lower expected return (0.14%) than the risk-free asset (0.36%). The investor borrows heavily in the recovery regime to invest heavily in small stocks but also in large stocks and bonds. The shares are unrealistic but they serve mainly the purpose of evaluating the accuracy of the proposed approximation. When the current regime is unknown, we assume that he uses the ergodic (long-run) probabilities of each regime. We observe that uncertainty significantly tame the portfolio shares. Given the uncertainty about the current regime, long-term bonds represent an important fraction of the portfolio.

[22] Adding a predictor would have created the need for a third grid in the numerical solution method. The complexity of considering 4 regimes and three assets poses already computational challenges. Similarly we did not include consumption in the problem since the problem without consumption ($\psi = \infty$) captures all the approximations we need to assess. Campani and Garcia (2019) show that the approximation needed when $\psi \neq 1$ in the Bellman equation is accurate.

5.3 Evaluating the accuracy of the approximate solution

Given the optimal solution obtained by simulation, we proceed to assess the accuracy of the proposed approximate analytical solution derived in section 4.2. Our main measure of accuracy will be the wealth equivalent utility loss created by using the approximate solution instead of the optimal solution obtained by simulation. We calculate this loss by considering two identical investors, one that follows the optimal strategy, the other the approximate strategy. The difference is that they do not invest the same initial amount. The investor who follows the approximate strategy begins with a wealth equal to $100. Given the two investors' strategies, we match their value functions (*i.e.*, utilities), so we can calculate the actual initial wealth the optimal investor needs to start with. Given that his strategy is optimal the wealth amount will be less than $100. The initial wealth difference will be the percentage wealth equivalent utility loss due to the suboptimal strategy. If this loss is negligible, we will conclude that the approximate analytical solutions provide accurate strategies.

In Table 3, the most right column reports the monthly wealth equivalent loss due to our approximate solution in comparison to the optimal strategy obtained through simulation. We compute it for all horizons and for all cases of known and unknown next period regime. When the next regime is assumed to be known the maximum loss was less than 0.02%. When the investor wants to account for a multi-regime economy but has no access to the filtered probabilities, the maximum loss is 0.06% when using long-run probabilities. These figures support the reliability of the analytical approximate solution proposed in this paper.

We complement this wealth loss criterion by showing the portfolio shares computed with the approximate analytical formulas. For the known next period cases, for all regimes, we can see that the shares are very close to the optimal ones. When we move to the case where the investor uses steady-state probabilities, the respective proportions and the long and short asset positions match, but not as closely as when the regime is known.

Overall we can conclude that the approximate solutions offer a very good accuracy and should help the investor who considers a multi-regime model of the economy. As we will see in the next section this accuracy becomes extremely important in a real-time out-of-sample dynamic portfolio allocation since it allows to compute the shares month after month given the estimated conditional probabilities without any difficulty. This is quite a gain compared to a numerical solution based on simulation.

5.4 The economic importance of regimes

In this section we will first evaluate whether hedging demands due to regime changes are important by considering a myopic investor who ignores these changes and measuring his wealth equivalent loss with respect to an investor who follows an optimal strategy. A second way to gauge the importance of the regimes is to conduct an out-of-sample exercise to measure the extra returns generated by considering regimes with respect to a single regime economy.

5.4.1 Are hedging demands important?

In Table 3 we saw that the shares invested in each asset class varied very little with the horizon. Therefore, hedging demands due to regime changes appear very small. We can therefore wonder whether hedging demands are really important to dynamic investors in the multi-regime economy without predictability. To address this issue, we assess the wealth equivalent loss one investor incurs when ignoring hedging demands. We use the Bellman equation (29) of a myopic investor who follows a myopic portfolio strategy (*i.e.*, without hedging demands):

$$\alpha_{\mathbf{t}} = \frac{1}{\gamma} \left(\mathbf{D}_{\mathbf{s,t}}\pi_{\mathbf{t}} - \mathbf{r}\mathbf{1}\right)^{\mathbf{T}} \left[\left(\mathbf{V}\pi_{\mathbf{t}}\right)\left(\mathbf{V}\pi_{\mathbf{t}}\right)^{\mathbf{T}}\right]^{-\mathbf{1}} \tag{28}$$

We follow the same steps as in the evaluation of the wealth loss due to the approximate solution with respect to the optimal one. The approximate solution is given by the same value function as before, but with different coefficients[23]. To find the total wealth equivalent loss, we equate the optimal value function (in which initial wealth is unknown) to the myopic sub-optimal one (in which wealth is standardized to 100) to solve for the unknown initial wealth of the optimal strategy. The difference to 100 gives the total percentage wealth equivalent loss due to the myopic strategy.

We computed the total wealth equivalent loss for the myopic strategy starting from each of the four regimes, from ergodic and from equal probabilities for horizons from 1 month to 10 years and the maximum loss was less than 0.3% per month[24]. We can safely conclude that the costs of ignoring hedging demands due to switches in regimes are very low, if not negligible. This conclusion is known in the literature (see Ang and Bekaert (2002) and Guidolin and Timmermann (2007)).

[23] These coefficients solve a similar (although simpler) system of equations, which we show in the online appendix. The non-predictability case there is nested when $\mathbb{B}$ is a zero column matrix.

[24] Even if we expand the horizon up to 40 years, this value raises only to 0.59%.

5.4.2 Out-of-sample performance

In this section we conduct a simple out-of-sample exercise to compare a four-regime model to a single-regime model and to a 1/N portfolio. The objective is to show that the approximate analytical formulas help considerably in conducting portfolio rebalancing in real time for a long-horizon investor. For simplicity, we do not re-estimate the parameters of the models at each point in time, so we give the investors some benefit of hindsight and the overall performance has an optimistic bias. However, the investor will use only past information in computing the conditional probabilities given the values of the parameters[25]. The main advantage of our approximate solution method in terms of computational time with respect to the numerical solution based on simulations is obvious. Given that the approximation appears to be reliable, it should be useful to portfolio managers.

Average performance results for an investor with a ten-year horizon and monthly rebalancing with allowed short-sales are reported in Table 4. For all sub-samples, the four-regime portfolio strategy results in an average return close to double the performance of the single-regime strategy. The latter dominates in turn the equal-weight portfolio. The variability of the four-regime portfolio returns is of course much higher, but overall its Sharpe ratio is slightly better than the single-regime. The cumulative returns over time are plotted in Figure 1 for the three sub-samples. All graphs show that the last 20 years were very fruitful for investors who accounted for regimes because of important disruptions such as the dot.com bubble and the 2008 financial crisis.

5.5 Including predictability

In this section, we include predictability as in the general model detailed in Section 3. Following the literature (*e.g.*, Guidolin and Timmermann (2007), Campbell and Viceira (1999) and Campbell et al. (2003) among others), we choose the dividend yield as predictor of asset returns (dividend yield over the Standard & Poors 500 Composite Index). We take the natural logarithm of the percentage values in order to make the predictor consistent with the process given by equation (3)[26]. The time period analyzed, all other time series and constants (β and the

[25] Implementing an actual procedure that investors can use requires to determine the frequency of estimation of the model, for example every month or every year, and to choose to use all past returns or a fixed window of a few years. Guidolin and Timmermann (2007) run a similar exercise by keeping all past information and accounting for the uncertainty attached to the parameter estimates.

[26] This means that the predictor data series should admit negative values given that its process is not exponential.

riskless rate for example) remain the same as in the case without predictability. To save space, we report the parameter estimation results with the predictor and the corresponding filtered probabilities in the online appendix[27].

5.5.1 Dynamic strategies with predictability and sensitivity to the predictor

To study the new features that predictability adds to the dynamic portfolio allocation problem, we start analyzing how the shares evolve as function of investor's horizon when the predictor is set at its overall sample average ($\overline{\text{p}}$). We include in Figure 2 the plots when the current regime is unknown[28]. As in the case without predictor the shares do not change much with the horizon and are positive for large and small stocks and for bonds. The shares are similar when the investor uses ergodic probabilities for assessing the likelihood of each regime.

The level of the predictor with which we compute the shares is of course key to the optimal portfolio strategies. Our approximate formulas allow us to compute easily the shares for all horizons when we set the predictor at different values. To illustrate the sensitivity of the results, we plot in Figures 3 and 4 the portfolio shares for horizons of 1 month to 10 years when the predictor is two standard deviations above and below its historical mean. We see that the shares vary more in both cases with the horizon.

Quite intuitively, the shares invested in stocks both large and small are increasing with the horizon when the dividend yield is above its mean and decreasing when it falls below. The difference between the short and the long horizon is more marked for small stocks than it is for large stocks, especially when the predictor is low. This is in line with Guidolin and Timmermann (2007). While both stocks and bonds have a positive share when the predictor is high and investors borrow at the risk-free rate, bonds serve as a hedge when the dividend yield is low.

5.5.2 Predictability and hedging demands

Using the same methodology as before, we calculate the expected wealth equivalent loss a dynamic myopic investor suffers in the beginning of the investment period for ignoring hedging demands. Results are reported in Table 5. If the magnitude of the losses has clearly increased with predictability, the importance of hedging demands will depend on the predictor's value,

[27] Overall, we note that the information provided by the predictor allows investors to make more accurate inferences about the regimes, but that predictability keeps the overall nature of the regimes and does not affect drastically the magnitude of the parameters with respect to the no-predictability case. The main difference is for the crash regime, which becomes less persistent and of a shorter duration.

[28] For completeness we include the graphs when the regime is known in the online appendix.

as suggested by the previous subsection. We observe that when the predictor is equal to its sample average, above or below its mean, the total loss is always less than 0.07% for a one-year horizon, which is negligible. The losses increase a bit for a 10-year horizon but remain below 1%. For a 40-year strategy, hedging demands become more significant, as the cost of being myopic represents losses that can reach more than 37% of the initial wealth, if the initial state of the economy is the persistent bull regime (regime 3). When the regime is unknown, the loss is close to 8% when the predictor value is high.

5.6 Including consumption

In this section, we determine how the presence of multiple regimes affects the consumption pattern of investors. We report in Figure 5 how optimal consumption evolves with and without predictability, for realistic values of ψ[29] and of the predictor[30].

In the upper panel, we show the consumption policy as a function of the horizon for each regime as well as when the investor uses ergodic or equal probabilities (the predictor is set at its historical mean, ψ is fixed at 0.75 and γ remains, as always in this paper, equal to 5). We can observe that consumption is roughly equal in both dynamic problems with and without predictability. Moreover, and somehow surprisingly, consumption policy barely changes with the regime.

The lower plots present the real-time exercise of how a 12-month horizon investor would consume with the actual filtered probabilities and predictor values from July 1988 to July 2018. In these graphs, we show five different investors: two willing to postpone consumption ($\psi = 1.25$ and $\psi = 1.5$), two others that hardly substitute consumption today for tomorrow ($\psi = 0.5$ and $\psi = 0.75$) and the myopic (in terms of consumption) investor ($\psi = 1$). These two plots are important because they use a broad real-time range for current probabilities and for the predictor. They confirm that consumption-to-wealth ratio variations with the predictor and with the regimes are very limited.

[29] For a similar problem without regimes Campani and Garcia (2019) analyze in detail the sensitivity of consumption and portfolio choices to the value of both preference parameters γ and ψ.

[30] We also analyzed the impact of including consumption on the portfolio strategy and concluded that it was small with and without predictability. Therefore, all previous results hold also with intermediate consumption.

6 Conclusion

This paper derives an approximate analytical solution to a dynamic portfolio and consumption problem under stochastic differential utility (recursive utility in continuous time) in a multi-regime economy. The conclusions provided by the models with and without predictability are in line with the paper by Guidolin and Timmermann (2007) that feature a portfolio allocation between large and small stocks, long-term government bonds and a riskless asset. However, we arrive at these results by using, instead of computationally demanding simulation methods, approximate formulas for the portfolio and consumption policies that were shown to be accurate. This enables dynamic investors to work with real-time allocation models under unobservable regimes and finite horizons. In other words, the implementation of our approximate method offers a practical solution to investors' problems.

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Appendices

A The Bellman Equation for the Portfolio Problem Without Consumption

With no intermediate consumption, the Bellmann equation simplifies to the following expression:

$$0 = \sup_{\{\alpha_t\}} \left[\begin{array}{c} -\beta J_t + \frac{\partial J_t}{\partial t} + J_w W_t \left[r + \alpha_t \left(\mathbf{D_{s,t}} \pi_t - r\mathbf{1} \right) \right] + \\ +\frac{1}{2} J_{ww} W_t^2 \alpha_t \left(\mathbf{V} \pi_t \right) \left(\mathbf{V} \pi_t \right)^{\mathbf{T}} \alpha_t^{\mathbf{T}} + J_p \mathbf{D_p} \pi_t + \frac{1}{2} J_{pp} \sigma_{\mathbf{p}} \sigma_{\mathbf{p}}^{\mathbf{T}} + \\ + \sum_{i=1}^{m} J_{\pi_i} \sum_{j=1}^{m} \lambda_{ji} \pi_{j,t} + \frac{1}{2} \sum_{i=1}^{m} J_{\pi_i \pi_i} \sigma_{\mathbf{i},\pi} \sigma_{\mathbf{i},\pi}^{\mathbf{T}} + J_{wp} W_t \alpha_t \left(\mathbf{V} \pi_t \right) \overline{\sigma}_{\mathbf{p}}^{\mathbf{T}} + \\ + \sum_{i=1}^{m} J_{w\pi_i} W_t \alpha_t \left(\mathbf{V} \pi_t \right) \overline{\sigma}_{\mathbf{i},\pi}^{\mathbf{T}} + \sum_{i=1}^{m} J_{p\pi_i} \sigma_{\mathbf{p}} \sigma_{\mathbf{i},\pi}^{\mathbf{T}} + \sum_{i<j} J_{\pi_i \pi_j} \sigma_{\mathbf{i},\pi} \sigma_{\mathbf{j},\pi}^{\mathbf{T}} \end{array} \right]. \tag{29}$$

Substituting the optimal expression for the portfolio weight, as given by equation (19), the final Bellman equation (without consumption) is given by:

$$\begin{aligned} 0 \; = \; & -\beta V_t + \frac{\partial V_t}{\partial t} + V_w W_t r + V_p \mathbf{D_p} \pi_t + \frac{\mathbf{1}}{\mathbf{2}} V_{pp} \sigma_{\mathbf{p}} \sigma_{\mathbf{p}}^{\mathbf{T}} + \\ & + \sum_{i=1}^{m} V_{p\pi_i} \sigma_{\mathbf{p}} \sigma_{\mathbf{i},\pi}^{\mathbf{T}} + \sum_{i,j=1}^{m} V_{\pi_i} \lambda_{ji} \pi_{j,t} + \frac{1}{2} \sum_{i,j=1}^{m} \left(V_{\pi_i \pi_j} \sigma_{\mathbf{i},\pi} \sigma_{\mathbf{j},\pi}^{\mathbf{T}} - \frac{V_{w\pi_i} V_{w\pi_j}}{V_{ww}} \overline{\sigma}_{\mathbf{i},\pi} \overline{\sigma}_{\mathbf{j},\pi}^{\mathbf{T}} \right) - \\ & - \frac{1}{2} \frac{V_w^2}{V_{ww}} \left(\mathbf{D_{s,t}} \pi_t - r\mathbf{1} \right)^{\mathbf{T}} \left[\left(\mathbf{V} \pi_t \right) \left(\mathbf{V} \pi_t \right)^{\mathbf{T}} \right]^{-\mathbf{1}} \left(\mathbf{D_{s,t}} \pi_t - r\mathbf{1} \right) - \frac{1}{2} \frac{V_{wp}^2}{V_{ww}} \overline{\sigma}_{\mathbf{p}} \overline{\sigma}_{\mathbf{p}}^{\mathbf{T}} - \\ & - \frac{V_w V_{wp}}{V_{ww}} \overline{\sigma}_{\mathbf{p}} \left(\mathbf{V} \pi_t \right)^{-\mathbf{1}} \left(\mathbf{D_{s,t}} \pi_t - r\mathbf{1} \right) - \sum_{i=1}^{m} \frac{V_w V_{w\pi_i}}{V_{ww}} \overline{\sigma}_{\mathbf{i},\pi} \left(\mathbf{V} \pi_t \right)^{-\mathbf{1}} \left(\mathbf{D_{s,t}} \pi_t - r\mathbf{1} \right) - \\ & - \sum_{i=1}^{m} \frac{V_{wp} V_{w\pi_i}}{V_{ww}} \overline{\sigma}_{\mathbf{p}} \overline{\sigma}_{\mathbf{i},\pi}^{\mathbf{T}}. \end{aligned} \tag{30}$$

B The Approximate Bellman Equation

To solve the Bellman equation, we need to make the volatility matrix $\mathbf{V}\pi_t$ constant. A natural way to do it is to take the unconditional expectation of Y_t, in other words to replace the transition probabilities π_t by the unconditional regime probabilities. We denote the corresponding volatility matrix by $\mathbf{V}\pi_\infty$.

The Bellman equation obtained with the approximate function $H(p_t, \pi_t, \tau)$ is as follows:

$$
\begin{aligned}
0 \;=\;& f(g) - A_0' - A_p' p_t - \sum_{i=1}^{m} A_i' \pi_{i,t} - B_p' p_t^2 - \sum_{i=1}^{m} B_i' \pi_{i,t}^2 - \sum_{i=1}^{m} C_{pi}' p_t \pi_{i,t} - \\
& - \sum_{i<j} C_{ij}' \pi_{i,t}\pi_{j,t} + (1-\gamma) r + \left(A_p + 2B_p p_t + \sum_{i=1}^{m} C_{pi}\pi_{i,t} \right) \mathbf{D_p} \pi_{\mathbf{t}} + \\
& + \frac{1}{2} \left[2B_p + \left(A_p + 2B_p p_t + \sum_{i=1}^{m} C_{pi}\pi_{i,t} \right)^2 \right] \sigma_{\mathbf{p}} \sigma_{\mathbf{p}}^{\mathbf{T}} + \\
& + \sum_{i=1}^{m} \left[C_{pi} + \left(A_p + 2B_p p_t + \sum_{k=1}^{m} C_{pk}\pi_{k,t} \right) \left(A_i + 2B_i \pi_{i,t} + C_{pi} p_t + \sum_{k\neq i} C_{ik}\pi_{k,t} \right) \right] \sigma_{\mathbf{p}} \sigma_{\mathbf{i},\pi}^{\mathbf{T}} + \\
& + \frac{1-\gamma}{\gamma} \sum_{i=1}^{m} \left(A_p + 2B_p p_t + \sum_{k=1}^{m} C_{pk}\pi_{k,t} \right) \left(A_i + 2B_i \pi_{i,t} + C_{pi} p_t + \sum_{k\neq i} C_{ik}\pi_{k,t} \right) \overline{\sigma}_{\mathbf{p}} \overline{\sigma}_{\mathbf{i},\pi}^{\mathbf{T}} + \\
& + \sum_{i,j=1}^{m} \lambda_{ji}\pi_{j,t} \left(A_i + 2B_i \pi_{i,t} + C_{pi} p_t + \sum_{k\neq i} C_{ik}\pi_{k,t} \right) + \sum_{i=1}^{m} B_i \sigma_{\mathbf{i},\pi} \sigma_{\mathbf{i},\pi}^{\mathbf{T}} + \sum_{i<j} C_{ij} \sigma_{\mathbf{i},\pi} \sigma_{\mathbf{j},\pi}^{\mathbf{T}} + \\
& + \frac{1}{2} \sum_{i,j=1}^{m} \left(A_i + 2B_i \pi_{i,t} + C_{pi} p_t + \sum_{k\neq i} C_{ik}\pi_{k,t} \right) \left(A_j + 2B_j \pi_{j,t} + C_{pj} p_t + \sum_{k\neq j} C_{jk}\pi_{k,t} \right) \sigma_{\mathbf{i},\pi} \sigma_{\mathbf{j},\pi}^{\mathbf{T}} + \\
& + \frac{1}{2} \frac{1-\gamma}{\gamma} \sum_{i,j=1}^{m} \left(A_i + 2B_i \pi_{i,t} + C_{pi} p_t + \sum_{k\neq i} C_{ik}\pi_{k,t} \right) \left(A_j + 2B_j \pi_{j,t} + C_{pj} p_t + \sum_{k\neq j} C_{jk}\pi_{k,t} \right) \overline{\sigma}_{\mathbf{i},\pi} \overline{\sigma}_{\mathbf{j},\pi}^{\mathbf{T}} + \\
& + \frac{1}{2} \frac{1-\gamma}{\gamma} \left(\mathbb{A}\pi_{\mathbf{t}} + \mathbb{B} p_t - r\mathbf{1} \right)^{\mathbf{T}} \left[\left(\mathbf{V}\pi_\infty \right) \left(\mathbf{V}\pi_\infty \right)^{\mathbf{T}} \right]^{-\mathbf{1}} \left(\mathbb{A}\pi_{\mathbf{t}} + \mathbb{B} p_t - r\mathbf{1} \right) + \\
& + \frac{1}{2} \frac{1-\gamma}{\gamma} \left(A_p + 2B_p p_t + \sum_{i=1}^{m} C_{pi}\pi_{i,t} \right)^2 \overline{\sigma}_{\mathbf{p}} \overline{\sigma}_{\mathbf{p}}^{\mathbf{T}} + \\
& + \frac{1-\gamma}{\gamma} \left(A_p + 2B_p p_t + \sum_{i=1}^{m} C_{pi}\pi_{i,t} \right) \overline{\sigma}_{\mathbf{p}} \left(\mathbf{V}\pi_\infty \right)^{-\mathbf{1}} \left(\mathbb{A}\pi_{\mathbf{t}} + \mathbb{B} p_t - r\mathbf{1} \right) + \\
& + \frac{1-\gamma}{\gamma} \sum_{i=1}^{m} \left(A_i + 2B_i \pi_{i,t} + C_{pi} p_t + \sum_{k\neq i} C_{ik}\pi_{k,t} \right) \overline{\sigma}_{\mathbf{i},\pi} \left(\mathbf{V}\pi_\infty \right)^{-\mathbf{1}} \left(\mathbb{A}\pi_{\mathbf{t}} + \mathbb{B} p_t - r\mathbf{1} \right).
\end{aligned}
\tag{31}
$$

For the problems with no consumption or with $\psi = 1$, this equation can be solved by matching equivalent terms, such that we find a system of $\frac{m^2}{2} + \frac{5m}{2} + 3$ equations (where m represents the number of regimes) with exactly the same number of (time-varying) unknowns (the coefficients of function g). The boundary condition for all coefficients is zero when $\tau = 0$. In the online appendix to this paper, we present the system of equations which can be solved with standard software. For the general problem in which $\psi \neq 1$, this equation is non-linear, what prevents us to solve it analytically. However, using the same log-linearization approximation technique explained in Campani and Garcia (2019), we are able to solve it. Details for this general solution can also be found in the online appendix. But when $\gamma = 1$, the solution to the system (31) implies that all coefficients are constant and equal to zero, except $A_0 = -\beta\tau$. As a consequence, the optimal portfolio is myopic (no hedging demands), as expected. Another nested case is the single state model: in such a case, $\pi_{\mathbf{i}}$ is constant and therefore $\sigma_{\mathbf{i},\pi}$ is zero (no hedging demands related to regime changes), leaving the investor with the same portfolio that a single state investor would see as optimal.

C Details on the Monte Carlo Simulation Procedure

We explain in this appendix the details of the Monte Carlo simulation procedure designed to assess the accuracy of our approximation technique. Starting with the dynamic problem, fix the starting point of the problem at $t = 0$, an horizon T, and a rebalancing frequency (which will be monthly). The investor's problem at any given moment t is[31].:

$$V\left(W_t, \widehat{\mathbf{Y}}_{\mathbf{t+1|t}}, t\right) = \sup_{\{\alpha_{\mathbf{t}}, \alpha_{\mathbf{t+1}}, \ldots, \alpha_{\mathbf{T-1}}\}} E_t\left[\frac{W_T^{1-\gamma}}{1-\gamma}\right], \qquad t = 0, 1, \ldots, T-1. \tag{32}$$

At date 0, the investor wants to find the optimal asset allocation $\alpha_{\mathbf{0}}$, given her horizon T, knowing that she will optimally reallocate again her resources monthly from $t = 1$ towards $t = T - 1$. The wealth constraint reads:

$$W_{t+1} = W_t\left[\left(1 - \alpha_{\mathbf{t}}^{\mathbf{T}}\mathbf{1}\right)e^r + \alpha_{\mathbf{t}}^{\mathbf{T}} e^{\overline{\mathbf{LR}}_{\mathbf{t}}}\right], \tag{33}$$

in which $\mathbf{e}^{\overline{\mathbf{LR}}_{\mathbf{t}}}$ is a column vector where its elements are the exponentials of the elements of the vector $\mathbf{LR}_{\mathbf{t}}$, defined in Section 5.1, without its last element (predictor change). With power utility, the value function can be written as:

$$V\left(W_t, \widehat{\mathbf{Y}}_{\mathbf{t+1|t}}, t\right) = \frac{W_t^{1-\gamma}}{1-\gamma} H\left(\widehat{\mathbf{Y}}_{\mathbf{t+1|t}}, t\right), \tag{34}$$

with terminal condition $H\left(\widehat{\mathbf{Y}}_{\mathbf{T+1|T}}, T\right) = 1$. Equations (32) and (34) imply the following recur-

[31] To be consistent with the notation used in this paper, observe that we use as state variable the filtered probabilities $\widehat{\mathbf{Y}}_{\mathbf{t+1|t}}$ as opposed to $\pi_{\mathbf{t}}$ because we are under a discrete-time model now. Recall that $\widehat{\mathbf{Y}}_{\mathbf{t+1|t}}$ represents the best guess of the probabilities to the next period, given the information available at the current period. The equations used to compute these probabilities can be found in the online appendix

sion:

$$H\left(\widehat{\mathbf{Y}}_{\mathbf{t+1|t}}, t\right) = \sup_{\{\alpha_t\}} E_t\left[\left(\frac{W_{t+1}}{W_t}\right)^{1-\gamma} H\left(\widehat{\mathbf{Y}}_{\mathbf{t+2|t+1}}, t+1\right)\right]. \quad (35)$$

This recursive equation is key to solve the problem using dynamic programming techniques. We start with the last investor decision, at time $t = T-1$: in this case we know that $H\left(\widehat{\mathbf{Y}}_{\mathbf{T+1|T}}, T\right) = 1$ for all $\widehat{\mathbf{Y}}_{\mathbf{T+1|T}}$ and the right-hand side of the recursion will only depend on wealth's return. The next step is to build a grid of possible values for the vector of regime probabilities ($\widehat{\mathbf{Y}}_{\mathbf{T|T-1}}$): Guidolin and Timmermann (2007) indicate in footnote 12 on page 3523 that five grid-discretization points (steps of 20%) guarantee sufficient accuracy in the calculation of optimal choices. To be on the conservative side we set our grid for each element of $\widehat{\mathbf{Y}}_{\mathbf{t+1|t}}$ with 8 discretization points $\{0, 0.125, 0.250, ..., 1\}$. For each point in the grid[32], we first simulate the next period regime (using the transition probability matrix) and, based on this simulated regime, simulate the corresponding risky assets returns. To be able to calculate the right-hand side of the recursion above, we do this simulation N times and use the law of large numbers to approximate the expectation. Guidolin and Timmermann (2007) indicate in the above referenced footnote that $N \geq 20,000$ is sufficient and use 30,000 in their paper. We follow them and use the same number of simulations.

To find this optimal allocation strategy, we again work with a grid for the portfolio weights $\alpha_{\mathbf{t}}$. We also set this grid with a 12,5% step for all assets. Having the right-hand side of the above recursion for every $\widehat{\mathbf{Y}}_{\mathbf{T|T-1}}$ in the grid, we know $H\left(\widehat{\mathbf{Y}}_{\mathbf{T|T-1}}, T-1\right)$ for every point in the probability grid. We repeat the process backwards until we reach $t = 0$, but at this point we know the value of $\widehat{\mathbf{Y}}_{\mathbf{1|0}}$ such that the optimal portfolio strategy at time 0, given the monthly rebalancings and the horizon T, is determined. One last detail remains to be explained: not at time $t = T-1$, but at all proceeding steps, given the grid for $\widehat{\mathbf{Y}}_{\mathbf{t+1|t}}$, nothing guarantees that $\widehat{\mathbf{Y}}_{\mathbf{t+2|t+1}}$ will also be in the grid. However we need the estimated values of $H\left(\widehat{\mathbf{Y}}_{\mathbf{t+2|t+1}}, t+1\right)$ but these values were calculated only for points in the grid. We set $\widetilde{\mathbf{Y}}_{\mathbf{t+2|t+1}}$ to the closest $\widehat{\mathbf{Y}}_{\mathbf{t+2|t+1}}$ in the probability grid (using the standard Euclidean distance).

[32] A point of the grid is a full vector $\widehat{\mathbf{Y}}_{\mathbf{t+1|t}}$.

D Tables

Table 1: Parameter Estimates of the Four-Regime Model. We report in Panel A the parameters and the optimal weights for the single state model (for $\gamma = 5$). The optimal weights for the single-state model sum up to 100% by short selling the riskless asset. Panel B presents the monthly parameter estimates for the four-state model from January 1953 to July 2018. Both panels do not allow for predictability. The volatilities are reported on the diagonals of the correlation/volatilities matrices. The average Treasury yield over the full period was estimated at 0.36% a month.

Panel A: Single State Model	Large Caps	Small Caps	LT Bonds
Mean Returns	0.92%	1.15%	0.49%
Correlation/Volatilities Matrix - Large Caps	4.08%		
Small Caps	73.94%	5.90%	
LT Bonds	8.76%	-3.67%	2.11%
Optimal Weights	33.39%	27.58%	46.95%
Panel B: Four State Model	Large Caps	Small Caps	LT Bonds
Mean Returns - Regime 1 (Crash)	-1.33%	-1.82%	0.71%
Regime 2 (Slow Growth)	0.92%	0.50%	0.44%
Regime 3 (Bull)	1.12%	1.49%	0.14%
Regime 4 (Recovery)	3.61%	6.53%	1.03%
Correlation/Volatilities Matrix - *Regime 1 (Crash)*			
Large Caps	6.09%		
Small Caps	76.04%	9.41%	
LT Bonds	-3.50%	-3.78%	2.59%
Correlation/Volatilities Matrix - *Regime 2 (Slow Growth)*			
Large Caps	2.91%		
Small Caps	73.34%	4.11%	
LT Bonds	12.25%	-4.15%	1.87%
Correlation/Volatilities Matrix - *Regime 3 (Bull)*			
Large Caps	3.25%		
Small Caps	73.19%	3.69%	
LT Bonds	2.79%	-1.49%	0.83%
Correlation/Volatilities Matrix - *Regime 4 (Recovery)*			
Large Caps	3.44%		
Small Caps	27.10%	3.64%	
LT Bonds	31.40%	-22.52%	3.30%

Transition Probabilities	Regime 1	Regime 2	Regime 3	Regime 4
Regime 1 (Crash)	77.48%	0.00%	0.00%	22.52%
Regime 2 (Slow Growth)	5.91%	91.44%	0.00%	2.65%
Regime 3 (Bull)	1.43%	0.00%	98.57%	0.00%
Regime 4 (Recovery)	8.40%	27.78%	3.68%	60.14%

Table 2: Probability Volatility Matrix and Ergodic Probabilities. We show below the estimated volatility matrix for the probability processes when there are 4 regimes, three risky assets and no predictability. The estimation procedure is explained in Section 5.1. Data are from January 1953 to July 2018. The table also reports the ergodic (unconditional) probabilities and the average duration of each regime.

	Z_1	Z_2	Z_3	Steady-state (ergodic) Probabilities	Average Duration
Regime 1	-5.36%	-1.86%	0.06%	17%	4 months
Regime 2	2.48%	-1.23%	-1.21%	40%	12 months
Regime 3	0.84%	0.92%	0.10%	31%	70 months
Regime 4	2.04%	2.17%	1.04%	12%	3 months

Table 3: Dynamic Strategies and Approximate Solution Accuracy We present the dynamic portfolio strategies for the four-state model without predictability, and the approximate solution accuracy measured by the wealth equivalent loss. The optimal shares are obtained numerically by simulation, while the approximate ones are computed with the approximate analytical formulas presented in the text. Results are reported for different horizons, and portfolios are unconstrained (*i.e.*, short-selling is allowed). We consider that the next period regime can be known (the investor sets the filtered probability $P(Y_{t+1} = i|\mathfrak{F}_t)$ to 1) or unknown (in which case the investor uses the steady-state probabilities of the regimes).

Horizon (months)	Dynamic Portfolio Strategy								Monthly Wealth Equivalent Loss
	Optimal shares				Approximate shares				
	Large Caps	Small Caps	LT Bonds	Risk Free	Large Caps	Small Caps	LT Bonds	Risk Free	
Regime 1 (Crash)									
1	-87.5%	-7.5%	100.0%	95.0%	-76.5%	-7.5%	99.7%	84.2%	0.00%
6	-75.0%	-7.5%	87.5%	95.0%	-74.0%	-7.1%	98.4%	82.7%	0.00%
12	-75.0%	-7.5%	100.0%	82.5%	-73.5%	-6.8%	98.3%	82.0%	0.00%
60	-75.0%	-7.5%	100.0%	82.5%	-73.7%	-6.3%	98.5%	81.5%	0.00%
120	-75.0%	-7.5%	87.5%	95.0%	-73.7%	-6.3%	98.5%	81.5%	0.00%
Regime 2 (Slow Growth)									
1	250.0%	-115.0%	-7.5%	-27.5%	252.5%	-114.7%	-7.6%	-30.1%	0.00%
6	250.0%	-115.0%	5.0%	-40.0%	251.6%	-115.5%	-7.9%	-28.2%	0.00%
12	262.5%	-115.0%	-7.5%	-40.0%	251.9%	-115.0%	-7.9%	-29.0%	0.00%
60	262.5%	-115.0%	-20.0%	-27.5%	251.5%	-113.7%	-7.6%	-30.3%	0.00%
120	262.5%	-115.0%	5.0%	-52.5%	251.5%	-113.5%	-7.5%	-30.4%	0.00%
Regime 3 (Bull)									
1	25.0%	147.5%	-600.0%	527.5%	23.6%	145.9%	-599.8%	530.3%	0.00%
6	25.0%	147.5%	-575.0%	502.5%	22.9%	140.9%	-604.1%	540.4%	0.01%
12	25.0%	147.5%	-575.0%	502.5%	23.6%	139.6%	-605.1%	542.0%	0.01%
60	12.5%	160.0%	-600.0%	527.5%	24.9%	135.9%	-607.3%	546.5%	0.01%
120	25.0%	147.5%	-612.5%	540.0%	25.1%	135.4%	-607.6%	547.2%	0.01%
Regime 4 (Recovery)									
1	192.5%	937.5%	262.5%	-1292.5%	180.2%	930.1%	291.0%	-1301.3%	0.00%
6	180.0%	925.0%	275.0%	-1280.0%	179.7%	923.8%	287.7%	-1291.2%	0.02%
12	180.0%	925.0%	287.5%	-1292.5%	180.6%	924.1%	287.6%	-1292.3%	0.01%
60	180.0%	912.5%	287.5%	-1280.0%	181.0%	925.0%	287.7%	-1293.7%	0.02%
120	180.0%	925.0%	287.5%	-1292.5%	181.1%	925.1%	287.8%	-1293.9%	0.01%
Steady-state (ergodic) Probabilities									
1	37.5%	50.0%	52.5%	-40.0%	35.3%	50.3%	64.0%	-49.6%	0.00%
6	50.0%	37.5%	27.5%	-15.0%	35.9%	48.2%	62.3%	-46.4%	0.02%
12	37.5%	37.5%	40.0%	-27.5%	36.4%	48.2%	62.1%	-46.7%	0.01%
60	62.5%	25.0%	52.5%	-40.0%	36.6%	48.0%	62.0%	-46.6%	0.02%
120	37.5%	25.0%	77.5%	-40.0%	36.6%	48.0%	61.9%	-46.6%	0.06%

Table 4: Out-of-sample Performance. We show below the out-of-sample performance in terms of monthly average returns, standard deviation and Sharpe ratio. We compared the four-regime model with the single-regime model and an equal-weight portfolio. To give a broader perspective of the performance over time, we start the out-of-sample exercise at different times, namely January 1960, January 1980 and January 2000. The sample ends in July 2018.

Accumulated Returns									
Windows	**Since 1960**			**Since 1980**			**Since 2000**		
Model	Four Regimes	Single Regime	Equal Weights	Four Regimes	Single Regime	Equal Weights	Four Regimes	Single Regime	Equal Weights
Average Returns	1.56%	0.83%	0.73%	1.83%	0.92%	0.78%	1.39%	0.63%	0.51%
Standard Deviation	7.94%	3.09%	2.48%	7.83%	3.01%	2.41%	6.31%	2.71%	2.29%
Sharpe Ratio	15.01%	14.75%	14.32%	18.90%	18.91%	17.73%	19.89%	18.50%	16.64%

Table 5: Expected Loss of a Myopic Investor. We show below the expected total cost of a myopic dynamic strategy when compared to the dynamic strategy with hedging demands as a function of the initial regime in the economy for three different horizons (given in months). The predictor value is set at its sample average and at this value plus two standard deviations. This cost corresponds to the expected wealth equivalent loss a myopic investor expects to suffer in the beginning of the investment period. Investor's relative risk aversion is set $\gamma = 5$.

Initial State	$p_t = \overline{p} - 2\delta_{p_t}$			$p_t = \overline{p}$			$p_t = \overline{p} + 2\delta_{p_t}$		
in Economy	T=12	T=120	T=480	T=12	T=120	T=480	T=12	T=120	T=480
Regime 1	0.02%	0.28%	9.43%	0.02%	0.48%	7.52%	0.04%	0.70%	2.77%
Regime 2	0.01%	0.00%	7.70%	0.00%	0.22%	6.01%	0.01%	0.94%	1.56%
Regime 3	0.07%	0.58%	32.68%	0.03%	0.07%	34.23%	0.00%	0.55%	37.58%
Regime 4	0.03%	0.48%	12.35%	0.01%	0.77%	10.51%	0.01%	0.31%	5.77%
Ergodic Prob.	0.03%	0.13%	1.75%	0.01%	0.24%	3.56%	0.01%	0.74%	7.94%

E Figures

Figure 1: Comparison of cumulative returns for three portfolio strategies: four-regime model, single-regime model, 1/N model over various sub-samples starting in 1960, 1980, 2000.

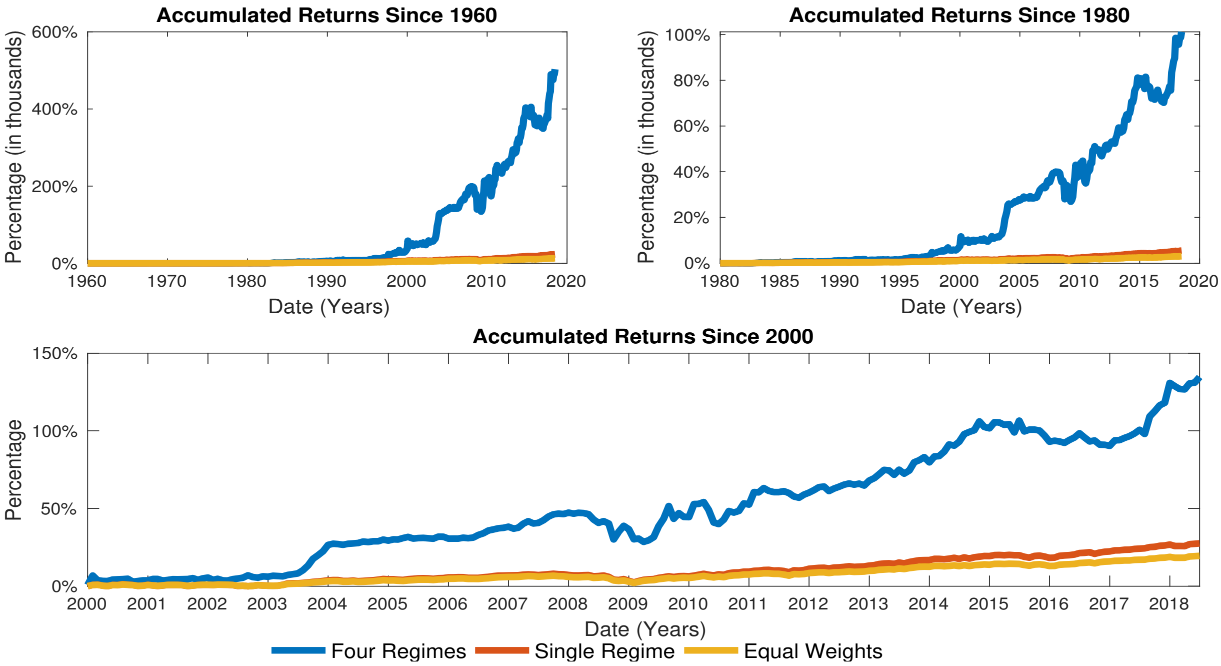

Figure 2: Dynamic strategies for different horizons in the model with predictability. The horizon is measured in years. The current regime is unknown and we plot the strategies when the current probabilities are either the ergodic ones or equal probabilities for each state. Investor's relative risk aversion is set $\gamma = 5$.

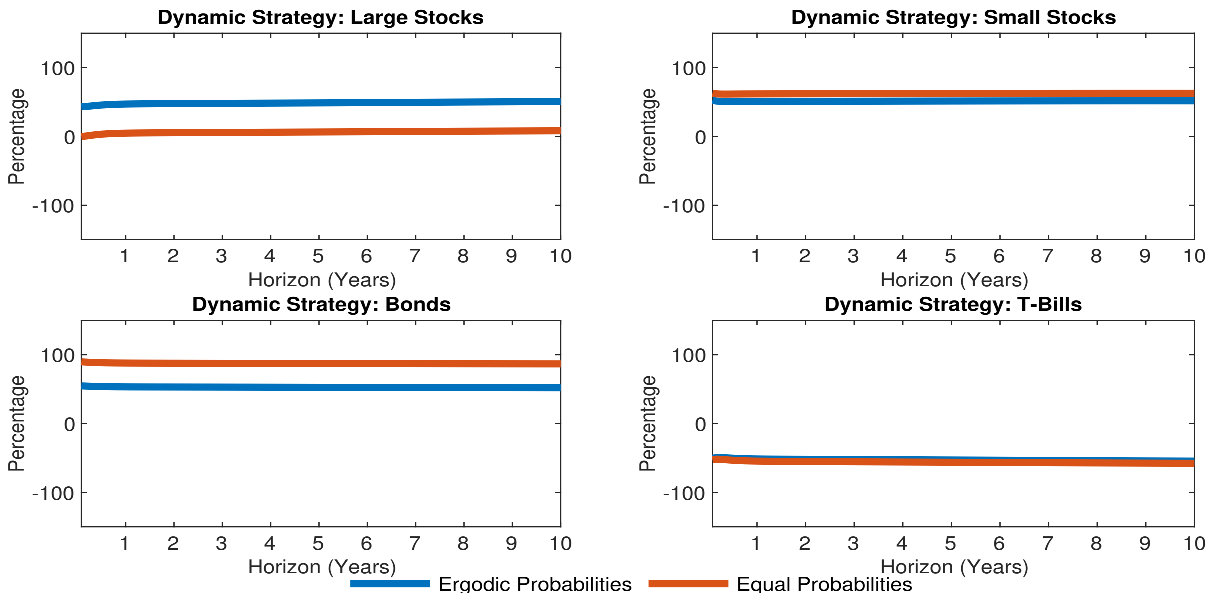

Figure 3: Dynamic strategies for different horizons in the model with predictability. The horizon is measured in years. The current regime is unknown and we plot below the strategies when the current probabilities are either the ergodic ones or equal probabilities for each state. Predictor's value is set two standard deviations above its mean. Investor's relative risk aversion is set $\gamma = 5$.

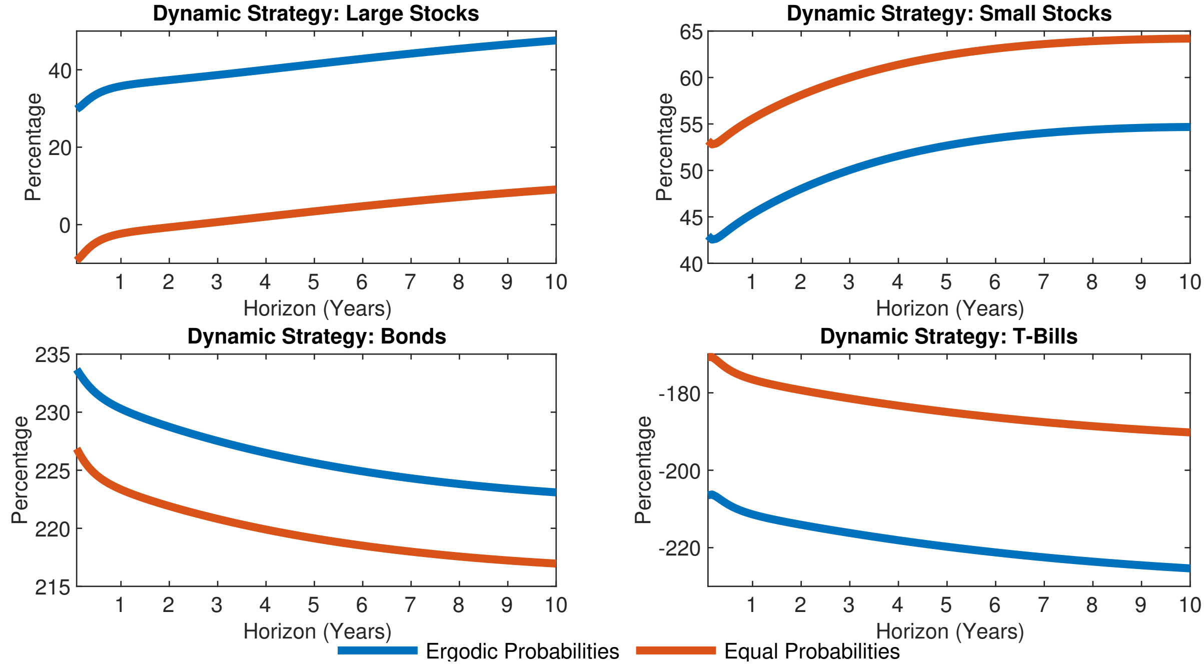

Figure 4: Dynamic strategies for different horizons in the model with predictability. The horizon is measured in years. The current regime is unknown and we plot below the strategies when the current probabilities are either the ergodic ones or equal probabilities for each state. Predictor's value is set two standard deviations below its mean. Investor's relative risk aversion is set $\gamma = 5$.

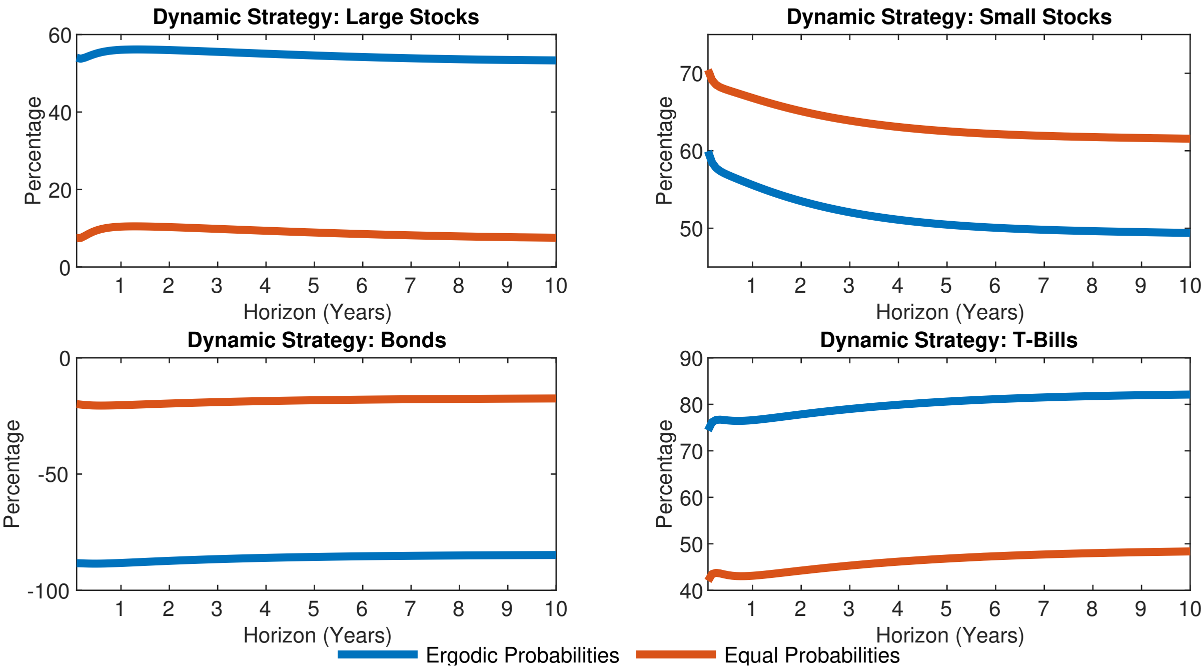

Figure 5: We plot below the consumption strategy for dynamic investors with $\gamma = 5$. In the upper plots, we draw six curves depending on the current regime for horizons from one month to 10 years ($\psi = 0.75$). In the lower plots, we show the consumption strategy followed by a 12-month horizon investor using real-time filtered probabilities (and predictor values) for the period from July, 1988 to July, 2018. The left plots are for the problem with no predictability while the ones in the right, for the model with predictability.