Online Appendix to the Paper: Optimal Portfolio Strategies in the Presence of Regimes in Asset Returns

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This document presents additional material related to the above paper.

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1 Estimation of Parameters

We assume that a monthly period, used in our paper, is short enough such that we can reasonably match Hamilton's transition probabilities with our densities of transition probabilities as follows[1]:

$$\text{Prob}\{Y_{t+1} = j | Y_t = i\} = P_{ij,\Delta t=1} = P_{ij} = \frac{\lambda_{ij}}{-\lambda_{ii}} \left(1 - e^{\lambda_{ii}}\right), i \neq j \tag{1}$$

in which we have conveniently chosen the time unit as the same as the period length (that is, one month). We will then have the following identities:

$$\lambda_{ii} = \ln P_{ii}, \text{ and} \tag{2a}$$

$$\lambda_{ij} = -\frac{P_{ij} \ln P_{ii}}{1 - P_{ii}}. \tag{2b}$$

We collect the (constant) discrete-time probabilities in a matrix we call $\mathbf{P}$:

$$\mathbf{P} = \begin{bmatrix} P_{11} & P_{12} & \dots & P_{1m} \\ P_{21} & P_{22} & \dots & P_{2m} \\ \dots & \dots & \dots & \dots \\ P_{m1} & P_{m2} & \dots & P_{mm} \end{bmatrix}. \tag{3}$$

The investor uses the filter explained in Hamilton (1989) to infer the current regime and take her optimal consumption and investment decisions. This optimal inference relies on iterating the pair of equations below:

$$\widehat{\mathbf{Y}}_{\mathbf{t|t}} = \frac{\widehat{\mathbf{Y}}_{\mathbf{t|t-1}} \circ \eta_{\mathbf{t}}}{\mathbf{1^T}\left(\widehat{\mathbf{Y}}_{\mathbf{t|t-1}} \circ \eta_{\mathbf{t}}\right)}, \text{ and} \tag{4a}$$

$$\widehat{\mathbf{Y}}_{\mathbf{t+1|t}} = \mathbf{P^T}\widehat{\mathbf{Y}}_{\mathbf{t|t}}, \tag{4b}$$

in which $\mathbf{1}$ represents here an mx1 vector of ones and the symbol $\circ$ denotes the element-by-element multiplication. $\widehat{\mathbf{Y}}_{\mathbf{t|t}}$ and $\widehat{\mathbf{Y}}_{\mathbf{t+1|t}}$ are also m x 1 vectors containing the updated probabilities of having each regime running respectively at times t and $t+1$, given the available information at time t. Finally, $\eta_{\mathbf{t}}$ is another m x 1 vector whose elements are the joint probability densities of the risky assets returns and predictor at time t, conditioned by being on each of

[1]Note that this would be exact if the period considered were the infinitesimally short period dt.

the states. The first equation above uses the current information (*i.e.*, the risky assets returns and predictor value) at time t to update the probabilities of each of the states in this last period t. The second equation then uses this update of the economy regime to optimally estimate the probabilities of being in each state in the next period (next month).

In order to put in practice the iteration above, we need a starting point. Obviously, this can be naturally guessed by the investor, based on her beliefs of the initial state of the economy. In this study, we will start from the long-run regime probabilities (often called ergodic or unconditional probabilities), which are given by:

$$\widehat{\mathbf{Y}}_{\mathbf{1|0}} = \left(A^T A\right)^{-1} A^T e_{m+1}, \tag{5}$$

where e_{m+1} denotes the last column vector of the identity matrix of order $m+1$ and A is an $(m+1)$ x m matrix in which the first m rows are the rows of $I_m - P^T$ (I_m is the identity matrix of order m) and the last row has only 1's[2]To find vector η_t, we recall that the processes followed by the assets and the predictor, if on a single-regime economy, admit the solutions below[3]:

$$\begin{bmatrix} \ln \frac{S_{1,t+1}}{S_{1,t}} \\ \ln \frac{S_{2,t+1}}{S_{2,t}} \\ \dots \\ \ln \frac{S_{n,t+1}}{S_{n,t}} \\ p_{t+1} - p_t \end{bmatrix} = \begin{bmatrix} \mu_{1,i} - \frac{1}{2}\sum\limits_{k=1}^{n} \sigma_{1k,i}^2 \\ \mu_{2,i} - \frac{1}{2}\sum\limits_{k=1}^{n} \sigma_{2k,i}^2 \\ \dots \\ \mu_{n,i} - \frac{1}{2}\sum\limits_{k=1}^{n} \sigma_{nk,i}^2 \\ \mu_{p,i} \end{bmatrix} + \begin{bmatrix} \sigma_{11,i} & 0 & \dots & 0 & 0 \\ \sigma_{21,i} & \sigma_{22,i} & \dots & 0 & 0 \\ \dots & \dots & \dots & 0 & 0 \\ \sigma_{n1,i} & \sigma_{n2,i} & \dots & \sigma_{nn,i} & 0 \\ \sigma_{p1} & \sigma_{p2} & \dots & \sigma_{pn} & \sigma_{pp} \end{bmatrix} \begin{bmatrix} \Delta Z_{1,t} \\ \Delta Z_{2,t} \\ \dots \\ \Delta Z_{n,t} \\ \Delta Z_{p,t} \end{bmatrix}, \tag{6}$$

such that the element of vector η_t occupying row i is[4]:

$$\eta_{t,i} = \frac{1}{(2\pi)^{\frac{n+1}{2}} |\sigma_i \sigma_i^T|^{\frac{1}{2}}} \exp\left[-\frac{1}{2}\left(\mathbf{LR_t} - \mathbf{MLR_i}\right)^T \left(\sigma_i \sigma_i^T\right)^{-1} \left(\mathbf{LR_t} - \mathbf{MLR_i}\right)\right], \tag{7}$$

where $\mathbf{LR_t}$ is an $(n+1)$ x 1 vector of the n risky assets observed log-returns at period t together with the predictor change in the same period (*i.e.*, the left-hand side of equation (6)). On its turn, $\mathbf{MLR_i}$ stores the mean of log-returns together with the mean predictor change conditioned

[2]The demonstration of this formula can be found at Hamilton (1989).

[3]We have assumed that any regime switch will take place at the end of period, as it is the standard in discrete-time regime switching models.

[4]Matrix σ_i is an $(n+1)$ x $(n+1)$ matrix built of $\sigma_{s,i}$ concatenated with σ_p in the last row and zeros in the first n rows of its last column, such as in equation (6). The expression $|\sigma_i {\sigma_i}^T|$ denotes the determinant of matrix $\sigma_i \sigma_i^T$.

on the regime:

$$\mathbf{LR_t} = \begin{bmatrix} \ln \frac{S_{1,t+1}}{S_{1,t}} \\ \ln \frac{S_{2,t+1}}{S_{2,t}} \\ \dots \\ \ln \frac{S_{n,t+1}}{S_{n,t}} \\ p_{t+1} - p_t \end{bmatrix} \quad \text{and} \quad \mathbf{MLR_i} = \begin{bmatrix} \mu_{1,i} - \frac{1}{2}\sum_{k=1}^{n} \sigma^2_{1k,i} \\ \mu_{2,i} - \frac{1}{2}\sum_{k=1}^{n} \sigma^2_{2k,i} \\ \dots \\ \mu_{n,i} - \frac{1}{2}\sum_{k=1}^{n} \sigma^2_{nk,i} \\ \mu_{p,i} \end{bmatrix}. \tag{8}$$

The parameter values are estimated using maximum likelihood estimation and the same methodology as in Hamilton (1989).

2 The System of Equations to the Problem with No Consumption

Equation a (independent term):[5]

$$\begin{aligned}
A_0' &= -\beta + (1-\gamma)\, r + \left(B_p + \frac{1}{2}A_p^2\right)\sigma_{\mathbf{p}}\sigma_{\mathbf{p}}^{\mathbf{T}} + \frac{1-\gamma}{2\gamma}A_p^2\overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{p}}^{\mathbf{T}} + \\
&+ \sum_{i=1}^{m}\left(C_{pi} + A_pA_i\right)\sigma_{\mathbf{p}}\sigma_{\mathbf{i},\pi}^{\mathbf{T}} + \frac{1-\gamma}{\gamma}A_p\sum_{i=1}^{m}A_i\overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{i},\pi}^{\mathbf{T}} + \\
&+ \sum_{i=1}^{m}B_i\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{i},\pi}^{\mathbf{T}} + \sum_{i<j}C_{ij}\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{j},\pi}^{\mathbf{T}} + \frac{1}{2}\sum_{i,j=1}^{m}A_iA_j\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{j},\pi}^{\mathbf{T}} + \\
&+ \frac{1-\gamma}{2\gamma}\sum_{i,j=1}^{m}A_iA_j\overline{\sigma}_{\mathbf{i},\pi}\overline{\sigma}_{\mathbf{j},\pi}^{\mathbf{T}} + \frac{1-\gamma}{2\gamma}r^2\mathbf{1}^{\mathbf{T}}\left[\left(\mathbf{V}\pi_\infty\right)\left(\mathbf{V}\pi_\infty\right)^{\mathbf{T}}\right]^{\mathbf{-1}}\mathbf{1} - \\
&- \frac{1-\gamma}{\gamma}rA_p\overline{\sigma}_{\mathbf{p}}\left(\mathbf{V}\pi_\infty\right)^{\mathbf{-1}}\mathbf{1} - \frac{1-\gamma}{\gamma}r\sum_{i=1}^{m}A_i\overline{\sigma}_{\mathbf{i},\pi}\left(\mathbf{V}\pi_\infty\right)^{\mathbf{-1}}\mathbf{1}.
\end{aligned} \tag{9a}$$

Equation b (p_t term):

$$\begin{aligned}
A_p' &= 2A_pB_p\sigma_{\mathbf{p}}\sigma_{\mathbf{p}}^{\mathbf{T}} + 2\frac{1-\gamma}{\gamma}A_pB_p\overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{p}}^{\mathbf{T}} + \sum_{i=1}^{m}\left(A_pC_{pi} + 2A_iB_p\right)\sigma_{\mathbf{p}}\sigma_{\mathbf{i},\pi}^{\mathbf{T}} + \\
&+ \frac{1-\gamma}{\gamma}\sum_{i=1}^{m}\left(A_pC_{pi} + 2A_iB_p\right)\overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{i},\pi}^{\mathbf{T}} + \frac{1}{2}\sum_{i,j=1}^{m}\left(A_iC_{pj} + A_jC_{pi}\right)\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{j},\pi}^{\mathbf{T}} + \\
&+ \frac{1-\gamma}{2\gamma}\sum_{i,j=1}^{m}\left(A_iC_{pj} + A_jC_{pi}\right)\overline{\sigma}_{\mathbf{i},\pi}\overline{\sigma}_{\mathbf{j},\pi}^{\mathbf{T}} - \frac{1-\gamma}{\gamma}r\mathbb{B}^{\mathbf{T}}\left[\left(\mathbf{V}\pi_\infty\right)\left(\mathbf{V}\pi_\infty\right)^{\mathbf{T}}\right]^{\mathbf{-1}}\mathbf{1} + \\
&+ \frac{1-\gamma}{\gamma}A_p\overline{\sigma}_{\mathbf{p}}\left(\mathbf{V}\pi_\infty\right)^{\mathbf{-1}}\mathbb{B} - 2\frac{1-\gamma}{\gamma}rB_p\overline{\sigma}_{\mathbf{p}}\left(\mathbf{V}\pi_\infty\right)^{\mathbf{-1}}\mathbf{1} + \\
&+ \frac{1-\gamma}{\gamma}\sum_{i=1}^{m}A_i\overline{\sigma}_{\mathbf{i},\pi}\left(\mathbf{V}\pi_\infty\right)^{\mathbf{-1}}\mathbb{B} - \frac{1-\gamma}{\gamma}r\sum_{i=1}^{m}C_{pi}\overline{\sigma}_{\mathbf{i},\pi}\left(\mathbf{V}\pi_\infty\right)^{\mathbf{-1}}\mathbf{1}.
\end{aligned} \tag{9b}$$

[5] In all the following equations, $\mathbb{A}_{:,i}$ represents the i^{th} column of matrix $\mathbb{A}$.

Equation c ($\pi_{i,t}$, $i \in R$, terms):

$$\begin{aligned}
A_i' \;=\;& A_p\mu_{p,i} + A_pC_{pi}\sigma_{\mathbf{p}}\sigma_{\mathbf{p}}^{\mathbf{T}} + \frac{1-\gamma}{\gamma}A_pC_{pi}\overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{p}}^{\mathbf{T}} + (2A_pB_i + A_iC_{pi})\,\sigma_{\mathbf{p}}\sigma_{\mathbf{i},\pi}^{\mathbf{T}} +\\
&+ \sum_{j\neq i}(A_pC_{ji} + A_jC_{pi})\,\sigma_{\mathbf{p}}\sigma_{\mathbf{j},\pi}^{\mathbf{T}} + \frac{1-\gamma}{\gamma}(2A_pB_i + A_iC_{pi})\,\overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{i},\pi}^{\mathbf{T}} +\\
&+ \frac{1-\gamma}{\gamma}\sum_{j\neq i}(A_pC_{ji} + A_jC_{pi})\,\overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{j},\pi}^{\mathbf{T}} + \sum_{j=1}^{m}\lambda_{ij}A_j + 2A_iB_i\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{i},\pi}^{\mathbf{T}} +\\
&+ \sum_{j\neq i}(A_iC_{ji} + 2A_jB_i)\,\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{j},\pi}^{\mathbf{T}} + \frac{1}{2}\sum_{j,k\neq i}(A_jC_{ki} + A_kC_{ji})\,\sigma_{\mathbf{j},\pi}\sigma_{\mathbf{k},\pi}^{\mathbf{T}} +\\
&+ 2\frac{1-\gamma}{\gamma}A_iB_i\overline{\sigma}_{\mathbf{i},\pi}\overline{\sigma}_{\mathbf{i},\pi}^{\mathbf{T}} + \frac{1-\gamma}{\gamma}\sum_{j\neq i}(A_iC_{ji} + 2A_jB_i)\,\overline{\sigma}_{\mathbf{i},\pi}\overline{\sigma}_{\mathbf{j},\pi}^{\mathbf{T}} +\\
&+ \frac{1-\gamma}{2\gamma}\sum_{j,k\neq i}(A_jC_{ki} + A_kC_{ji})\,\overline{\sigma}_{\mathbf{j},\pi}\overline{\sigma}_{\mathbf{k},\pi}^{\mathbf{T}} - \frac{1-\gamma}{\gamma}r\mathbf{1}^{\mathbf{T}}\left[(\mathbf{V}\pi_\infty)(\mathbf{V}\pi_\infty)^{\mathbf{T}}\right]^{-\mathbf{1}}\mathbb{A}_{:,i} +\\
&+ \frac{1-\gamma}{\gamma}\left[A_p\overline{\sigma}_{\mathbf{p}}(\mathbf{V}\pi_\infty)^{-\mathbf{1}}\mathbb{A}_{:,i} - rC_{pi}\overline{\sigma}_{\mathbf{p}}(\mathbf{V}\pi_\infty)^{-\mathbf{1}}\mathbf{1} + \sum_{j=1}^{m}A_j\overline{\sigma}_{\mathbf{j},\pi}(\mathbf{V}\pi_\infty)^{-\mathbf{1}}\mathbb{A}_{:,i}\right] +\\
&- 2\frac{1-\gamma}{\gamma}rB_i\overline{\sigma}_{\mathbf{i},\pi}(\mathbf{V}\pi_\infty)^{-\mathbf{1}}\mathbf{1} - \frac{1-\gamma}{\gamma}r\sum_{j\neq i}C_{ji}\overline{\sigma}_{\mathbf{j},\pi}(\mathbf{V}\pi_\infty)^{-\mathbf{1}}\mathbf{1}.
\end{aligned} \tag{9c}$$

Equation d (p_t^2 term):

$$\begin{aligned}
B_p' \;=\;& 2B_p^2\sigma_{\mathbf{p}}\sigma_{\mathbf{p}}^{\mathbf{T}} + 2\sum_{i=1}^{m}B_pC_{pi}\sigma_{\mathbf{p}}\sigma_{\mathbf{i},\pi}^{\mathbf{T}} + 2\frac{1-\gamma}{\gamma}\sum_{i=1}^{m}B_pC_{pi}\overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{i},\pi}^{\mathbf{T}} +\\
&+ \frac{1}{2}\sum_{i,j=1}^{m}C_{pi}C_{pj}\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{j},\pi}^{\mathbf{T}} + \frac{1-\gamma}{2\gamma}\sum_{i,j=1}^{m}C_{pi}C_{pj}\overline{\sigma}_{\mathbf{i},\pi}\overline{\sigma}_{\mathbf{j},\pi}^{\mathbf{T}} +\\
&+ \frac{1-\gamma}{2\gamma}\mathbb{B}^{\mathbf{T}}\left[(\mathbf{V}\pi_\infty)(\mathbf{V}\pi_\infty)^{\mathbf{T}}\right]^{-\mathbf{1}}\mathbb{B} + 2\frac{1-\gamma}{\gamma}B_p^2\overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{p}}^{\mathbf{T}} +\\
&+ 2\frac{1-\gamma}{\gamma}B_p\overline{\sigma}_{\mathbf{p}}(\mathbf{V}\pi_\infty)^{-\mathbf{1}}\mathbb{B} + \frac{1-\gamma}{\gamma}\sum_{i=1}^{m}C_{pi}\overline{\sigma}_{\mathbf{i},\pi}(\mathbf{V}\pi_\infty)^{-\mathbf{1}}\mathbb{B}.
\end{aligned} \tag{9d}$$

Equation e ($\pi_{i,t}^2$, $i \in R$, terms):

$$
\begin{aligned}
B_i' \;=\;& C_{pi}\mu_{p,i} + \frac{1}{2}C_{pi}^2\sigma_{\mathbf{p}}\sigma_{\mathbf{p}}^{\mathbf{T}} + \frac{1-\gamma}{2\gamma}C_{pi}^2\overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{p}}^{\mathbf{T}} + 2C_{pi}B_i\sigma_{\mathbf{p}}\sigma_{\mathbf{i},\pi}^{\mathbf{T}} + \\
&+ 2\frac{1-\gamma}{\gamma}C_{pi}B_i\overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{i},\pi}^{\mathbf{T}} + \sum_{j\neq i}C_{pi}C_{ji}\sigma_{\mathbf{p}}\sigma_{\mathbf{j},\pi}^{\mathbf{T}} + \frac{1-\gamma}{\gamma}\sum_{j\neq i}C_{pi}C_{ji}\overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{j},\pi}^{\mathbf{T}} + \\
&+ 2\lambda_{ii}B_i + \sum_{j\neq i}\lambda_{ij}C_{ji} + 2B_i^2\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{i},\pi}^{\mathbf{T}} + 2\frac{1-\gamma}{\gamma}B_i^2\overline{\sigma}_{\mathbf{i},\pi}\overline{\sigma}_{\mathbf{i},\pi}^{\mathbf{T}} + \\
&+ 2B_i\sum_{j\neq i}C_{ji}\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{j},\pi}^{\mathbf{T}} + 2\frac{1-\gamma}{\gamma}B_i\sum_{j\neq i}C_{ji}\overline{\sigma}_{\mathbf{i},\pi}\overline{\sigma}_{\mathbf{j},\pi}^{\mathbf{T}} + \frac{1}{2}\sum_{j,k\neq i}C_{ji}C_{ki}\sigma_{\mathbf{j},\pi}\sigma_{\mathbf{k},\pi}^{\mathbf{T}} + \\
&+ \frac{1-\gamma}{2\gamma}\sum_{j,k\neq i}C_{ji}C_{ki}\overline{\sigma}_{\mathbf{j},\pi}\overline{\sigma}_{\mathbf{k},\pi}^{\mathbf{T}} + \frac{1-\gamma}{2\gamma}\mathbb{A}_{:,i}^{\mathbf{T}}\left[\left(\mathbf{V}\pi_\infty\right)\left(\mathbf{V}\pi_\infty\right)^{\mathbf{T}}\right]^{-\mathbf{1}}\mathbb{A}_{:,i} + \\
&+ \frac{1-\gamma}{\gamma}C_{pi}\overline{\sigma}_{\mathbf{p}}\left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}}\mathbb{A}_{:,i} + 2\frac{1-\gamma}{\gamma}B_i\overline{\sigma}_{\mathbf{i},\pi}\left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}}\mathbb{A}_{:,i} + \\
&+ \frac{1-\gamma}{\gamma}\sum_{j\neq i}C_{ji}\overline{\sigma}_{\mathbf{j},\pi}\left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}}\mathbb{A}_{:,i}.
\end{aligned}
\tag{9e}
$$

Equation f ($p_t\pi_{i,t}$, $i \in R$, terms):

$$
\begin{aligned}
C_{pi}' \;=\;& 2B_p\mu_{p,i} + 2B_pC_{pi}\sigma_{\mathbf{p}}\sigma_{\mathbf{p}}^{\mathbf{T}} + 2\frac{1-\gamma}{\gamma}B_pC_{pi}\overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{p}}^{\mathbf{T}} + 4B_pB_i\sigma_{\mathbf{p}}\sigma_{\mathbf{i},\pi}^{\mathbf{T}} + C_{pi}^2\sigma_{\mathbf{p}}\sigma_{\mathbf{i},\pi}^{\mathbf{T}} + \\
&+ \sum_{j\neq i}\left(2B_pC_{ji} + C_{pi}C_{pj}\right)\sigma_{\mathbf{p}}\sigma_{\mathbf{j},\pi}^{\mathbf{T}} + 4\frac{1-\gamma}{\gamma}B_pB_i\overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{i},\pi}^{\mathbf{T}} + \frac{1-\gamma}{\gamma}C_{pi}^2\overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{i},\pi}^{\mathbf{T}} + \\
&+ \frac{1-\gamma}{\gamma}\sum_{j\neq i}\left(2B_pC_{ji} + C_{pi}C_{pj}\right)\overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{j},\pi}^{\mathbf{T}} + \sum_{j=1}^{m}\lambda_{ij}C_{pj} + 2B_iC_{pi}\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{i},\pi}^{\mathbf{T}} + \\
&+ 2\frac{1-\gamma}{\gamma}B_iC_{pi}\overline{\sigma}_{\mathbf{i},\pi}\overline{\sigma}_{\mathbf{i},\pi}^{\mathbf{T}} + \sum_{j\neq i}\left(2B_iC_{pj} + C_{pi}C_{ji}\right)\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{j},\pi}^{\mathbf{T}} + \\
&+ \frac{1-\gamma}{\gamma}\sum_{j\neq i}\left(2B_iC_{pj} + C_{pi}C_{ji}\right)\overline{\sigma}_{\mathbf{i},\pi}\overline{\sigma}_{\mathbf{j},\pi}^{\mathbf{T}} + \frac{1}{2}\sum_{j,k\neq i}\left(C_{pj}C_{ki} + C_{pk}C_{ji}\right)\sigma_{\mathbf{j},\pi}\sigma_{\mathbf{k},\pi}^{\mathbf{T}} + \\
&\frac{1-\gamma}{2\gamma}\sum_{j,k\neq i}\left(C_{pj}C_{ki} + C_{pk}C_{ji}\right)\overline{\sigma}_{\mathbf{j},\pi}\overline{\sigma}_{\mathbf{k},\pi}^{\mathbf{T}} + \frac{1-\gamma}{\gamma}\left(\mathbb{A}_{:,i}\right)^{\mathbf{T}}\left[\left(\mathbf{V}\pi_\infty\right)\left(\mathbf{V}\pi_\infty\right)^{\mathbf{T}}\right]^{-\mathbf{1}}\mathbb{B} + \\
&+ \frac{1-\gamma}{\gamma}\left[2B_p\overline{\sigma}_{\mathbf{p}}\left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}}\mathbb{A}_{:,i} + C_{pi}\overline{\sigma}_{\mathbf{p}}\left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}}\mathbb{B} + \sum_{j=1}^{m}C_{pj}\overline{\sigma}_{\mathbf{j},\pi}\left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}}\mathbb{A}_{:,i}\right] + \\
&+ 2\frac{1-\gamma}{\gamma}B_i\overline{\sigma}_{\mathbf{i},\pi}\left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}}\mathbb{B} + \frac{1-\gamma}{\gamma}\sum_{j\neq i}C_{ji}\overline{\sigma}_{\mathbf{j},\pi}\left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}}\mathbb{B}.
\end{aligned}
\tag{9f}
$$

Equation g ($\pi_{i,t}\pi_{j,t}$, $i < j \in R$, terms):

$$
\begin{aligned}
C'_{ij} \;=\; & C_{pi}\mu_{p,j} + C_{pj}\mu_{p,i} + C_{pi}C_{pj}\sigma_{\mathbf{p}}\sigma_{\mathbf{p}}^{\mathbf{T}} + \frac{1-\gamma}{\gamma}C_{pi}C_{pj}\sigma_{\mathbf{p}}\overline{\sigma}_{\mathbf{p}}^{\mathbf{T}} + \\
& + \left(2B_iC_{pj} + C_{pi}C_{ij}\right)\sigma_{\mathbf{p}}\sigma_{\mathbf{i},\pi}^{\mathbf{T}} + \left(2B_jC_{pi} + C_{pj}C_{ij}\right)\sigma_{\mathbf{p}}\sigma_{\mathbf{j},\pi}^{\mathbf{T}} + \\
& + \frac{1-\gamma}{\gamma}\left[\left(2B_iC_{pj} + C_{pi}C_{ij}\right)\overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{i},\pi}^{\mathbf{T}} + \left(2B_jC_{pi} + C_{pj}C_{ij}\right)\overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{j},\pi}^{\mathbf{T}}\right] + \\
& + \sum_{k\neq i,j}\left(C_{pi}C_{kj} + C_{pj}C_{ki}\right)\sigma_{\mathbf{p}}\sigma_{\mathbf{k},\pi}^{\mathbf{T}} + \frac{1-\gamma}{\gamma}\sum_{k\neq i,j}\left(C_{pi}C_{kj} + C_{pj}C_{ki}\right)\overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{k},\pi}^{\mathbf{T}} + \\
& + 2\lambda_{ij}B_j + 2\lambda_{ji}B_i + \lambda_{ii}C_{ij} + \lambda_{jj}C_{ij} + \sum_{k\neq i,j}\left(\lambda_{ik}C_{kj} + \lambda_{jk}C_{ki}\right) + \\
& + 2B_iC_{ij}\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{i},\pi}^{\mathbf{T}} + 2B_jC_{ij}\sigma_{\mathbf{j},\pi}\sigma_{\mathbf{j},\pi}^{\mathbf{T}} + 2\frac{1-\gamma}{\gamma}\left[B_iC_{ij}\overline{\sigma}_{\mathbf{i},\pi}\overline{\sigma}_{\mathbf{i},\pi}^{\mathbf{T}} + B_jC_{ij}\overline{\sigma}_{\mathbf{j},\pi}\overline{\sigma}_{\mathbf{j},\pi}^{\mathbf{T}}\right] + \\
& + \left(4B_iB_j + C_{ij}^2\right)\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{j},\pi}^{\mathbf{T}} + \frac{1-\gamma}{\gamma}\left(4B_iB_j + C_{ij}^2\right)\overline{\sigma}_{\mathbf{i},\pi}\overline{\sigma}_{\mathbf{j},\pi}^{\mathbf{T}} + \\
& + \sum_{k\neq i,j}\left(2B_iC_{jk} + C_{ij}C_{ki}\right)\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{k},\pi}^{\mathbf{T}} + \sum_{k\neq i,j}\left(2B_jC_{ik} + C_{ji}C_{kj}\right)\sigma_{\mathbf{j},\pi}\sigma_{\mathbf{k},\pi}^{\mathbf{T}} + \\
& + \frac{1-\gamma}{\gamma}\left[\sum_{k\neq i,j}\left(2B_iC_{jk} + C_{ij}C_{ki}\right)\overline{\sigma}_{\mathbf{i},\pi}\overline{\sigma}_{\mathbf{k},\pi}^{\mathbf{T}} + \sum_{k\neq i,j}\left(2B_jC_{ik} + C_{ji}C_{kj}\right)\overline{\sigma}_{\mathbf{j},\pi}\overline{\sigma}_{\mathbf{k},\pi}^{\mathbf{T}}\right] + \\
& + \sum_{k<l\neq i,j}\left(C_{ki}C_{lj} + C_{kj}C_{li}\right)\sigma_{\mathbf{k},\pi}\sigma_{\mathbf{l},\pi}^{\mathbf{T}} + \sum_{k\neq i,j}C_{ik}C_{jk}\sigma_{\mathbf{k},\pi}\sigma_{\mathbf{k},\pi}^{\mathbf{T}} + \\
& + \frac{1-\gamma}{\gamma}\left[\sum_{k<l\neq i,j}\left(C_{ki}C_{lj} + C_{kj}C_{li}\right)\overline{\sigma}_{\mathbf{k},\pi}\overline{\sigma}_{\mathbf{l},\pi}^{\mathbf{T}} + \sum_{k\neq i,j}C_{ik}C_{jk}\overline{\sigma}_{\mathbf{k},\pi}\overline{\sigma}_{\mathbf{k},\pi}^{\mathbf{T}}\right] + \\
& + \frac{1-\gamma}{\gamma}\left(\mathbb{A}_{:,i}\right)^{\mathbf{T}}\left[\left(\mathbf{V}\pi_\infty\right)\left(\mathbf{V}\pi_\infty\right)^{\mathbf{T}}\right]^{-\mathbf{1}}\mathbb{A}_{:,j} + \\
& + \frac{1-\gamma}{\gamma}C_{pi}\overline{\sigma}_{\mathbf{p}}\left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}}\mathbb{A}_{:,j} + \frac{1-\gamma}{\gamma}C_{pj}\overline{\sigma}_{\mathbf{p}}\left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}}\mathbb{A}_{:,i} + \\
& + 2\frac{1-\gamma}{\gamma}B_i\overline{\sigma}_{\mathbf{i},\pi}\left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}}\mathbb{A}_{:,j} + \frac{1-\gamma}{\gamma}\sum_{k\neq i}C_{ik}\overline{\sigma}_{\mathbf{k},\pi}\left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}}\mathbb{A}_{:,j} + \\
& + 2\frac{1-\gamma}{\gamma}B_j\overline{\sigma}_{\mathbf{j},\pi}\left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}}\mathbb{A}_{:,i} + \frac{1-\gamma}{\gamma}\sum_{k\neq j}C_{jk}\overline{\sigma}_{\mathbf{k},\pi}\left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}}\mathbb{A}_{:,i}.
\end{aligned}
\tag{9g}
$$

3 The System of Equations to the Problem when $\psi = 1$

Equation a (independent term):

$$
\begin{aligned}
A_0' \;=\; & -\beta A_0 + (1-\gamma)\,\beta\,(\ln\beta - 1) + (1-\gamma)\,r + \\
& + \left(B_p + \frac{1}{2}A_p^2\right)\sigma_{\mathbf{p}}\sigma_{\mathbf{p}}^{\mathbf{T}} + \frac{1-\gamma}{2\gamma}A_p^2\overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{p}}^{\mathbf{T}} + \\
& + \sum_{i=1}^{m}\left(C_{pi} + A_pA_i\right)\sigma_{\mathbf{p}}\sigma_{\mathbf{i},\pi}^{\mathbf{T}} + \frac{1-\gamma}{\gamma}A_p\sum_{i=1}^{m}A_i\overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{i},\pi}^{\mathbf{T}} + \\
& + \sum_{i=1}^{m}B_i\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{i},\pi}^{\mathbf{T}} + \sum_{i<j}C_{ij}\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{j},\pi}^{\mathbf{T}} + \frac{1}{2}\sum_{i,j=1}^{m}A_iA_j\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{j},\pi}^{\mathbf{T}} + \\
& + \frac{1-\gamma}{2\gamma}\sum_{i,j=1}^{m}A_iA_j\overline{\sigma}_{\mathbf{i},\pi}\overline{\sigma}_{\mathbf{j},\pi}^{\mathbf{T}} + \frac{1-\gamma}{2\gamma}r^2\mathbf{1}^{\mathbf{T}}\left[\left(\mathbf{V}\pi_\infty\right)\left(\mathbf{V}\pi_\infty\right)^{\mathbf{T}}\right]^{-\mathbf{1}}\mathbf{1} - \\
& - \frac{1-\gamma}{\gamma}rA_p\overline{\sigma}_{\mathbf{p}}\left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}}\mathbf{1} - \frac{1-\gamma}{\gamma}r\sum_{i=1}^{m}A_i\overline{\sigma}_{\mathbf{i},\pi}\left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}}\mathbf{1}.
\end{aligned}
\tag{10a}
$$

Equation b (p_t term):

$$
\begin{aligned}
A_p' \;=\; & -\beta A_p + 2A_pB_p\sigma_{\mathbf{p}}\sigma_{\mathbf{p}}^{\mathbf{T}} + 2\frac{1-\gamma}{\gamma}A_pB_p\overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{p}}^{\mathbf{T}} + \sum_{i=1}^{m}\left(A_pC_{pi} + 2A_iB_p\right)\sigma_{\mathbf{p}}\sigma_{\mathbf{i},\pi}^{\mathbf{T}} + \\
& + \frac{1-\gamma}{\gamma}\sum_{i=1}^{m}\left(A_pC_{pi} + 2A_iB_p\right)\overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{i},\pi}^{\mathbf{T}} + \frac{1}{2}\sum_{i,j=1}^{m}\left(A_iC_{pj} + A_jC_{pi}\right)\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{j},\pi}^{\mathbf{T}} + \\
& + \frac{1-\gamma}{2\gamma}\sum_{i,j=1}^{m}\left(A_iC_{pj} + A_jC_{pi}\right)\overline{\sigma}_{\mathbf{i},\pi}\overline{\sigma}_{\mathbf{j},\pi}^{\mathbf{T}} - \frac{1-\gamma}{\gamma}r\mathbb{B}^{\mathbf{T}}\left[\left(\mathbf{V}\pi_\infty\right)\left(\mathbf{V}\pi_\infty\right)^{\mathbf{T}}\right]^{-\mathbf{1}}\mathbf{1} + \\
& + \frac{1-\gamma}{\gamma}A_p\overline{\sigma}_{\mathbf{p}}\left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}}\mathbb{B} - 2\frac{1-\gamma}{\gamma}rB_p\overline{\sigma}_{\mathbf{p}}\left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}}\mathbf{1} + \\
& + \frac{1-\gamma}{\gamma}\sum_{i=1}^{m}A_i\overline{\sigma}_{\mathbf{i},\pi}\left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}}\mathbb{B} - \frac{1-\gamma}{\gamma}r\sum_{i=1}^{m}C_{pi}\overline{\sigma}_{\mathbf{i},\pi}\left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}}\mathbf{1}.
\end{aligned}
\tag{10b}
$$

Equation c ($\pi_{i,t}$, $i \in R$, terms):

$$\begin{aligned}
A_i' \;=\; & -\beta A_i + A_p \mu_{p,i} + A_p C_{pi} \sigma_{\mathbf{p}} \sigma_{\mathbf{p}}^{\mathbf{T}} + \frac{1-\gamma}{\gamma} A_p C_{pi} \overline{\sigma}_{\mathbf{p}} \overline{\sigma}_{\mathbf{p}}^{\mathbf{T}} + \left(2A_p B_i + A_i C_{pi}\right) \sigma_{\mathbf{p}} \sigma_{\mathbf{i},\pi}^{\mathbf{T}} + \\
& + \sum_{j \neq i} \left(A_p C_{ji} + A_j C_{pi}\right) \sigma_{\mathbf{p}} \sigma_{\mathbf{j},\pi}^{\mathbf{T}} + \frac{1-\gamma}{\gamma} \left(2A_p B_i + A_i C_{pi}\right) \overline{\sigma}_{\mathbf{p}} \overline{\sigma}_{\mathbf{i},\pi}^{\mathbf{T}} + \\
& + \frac{1-\gamma}{\gamma} \sum_{j \neq i} \left(A_p C_{ji} + A_j C_{pi}\right) \overline{\sigma}_{\mathbf{p}} \overline{\sigma}_{\mathbf{j},\pi}^{\mathbf{T}} + \sum_{j=1}^{m} \lambda_{ij} A_j + 2 A_i B_i \sigma_{\mathbf{i},\pi} \sigma_{\mathbf{i},\pi}^{\mathbf{T}} + \\
& + \sum_{j \neq i} \left(A_i C_{ji} + 2A_j B_i\right) \sigma_{\mathbf{i},\pi} \sigma_{\mathbf{j},\pi}^{\mathbf{T}} + \frac{1}{2} \sum_{j,k \neq i} \left(A_j C_{ki} + A_k C_{ji}\right) \sigma_{\mathbf{j},\pi} \sigma_{\mathbf{k},\pi}^{\mathbf{T}} + \\
& + 2 \frac{1-\gamma}{\gamma} A_i B_i \overline{\sigma}_{\mathbf{i},\pi} \overline{\sigma}_{\mathbf{i},\pi}^{\mathbf{T}} + \frac{1-\gamma}{\gamma} \sum_{j \neq i} \left(A_i C_{ji} + 2A_j B_i\right) \overline{\sigma}_{\mathbf{i},\pi} \overline{\sigma}_{\mathbf{j},\pi}^{\mathbf{T}} + \\
& + \frac{1-\gamma}{2\gamma} \sum_{j,k \neq i} \left(A_j C_{ki} + A_k C_{ji}\right) \overline{\sigma}_{\mathbf{j},\pi} \overline{\sigma}_{\mathbf{k},\pi}^{\mathbf{T}} - \frac{1-\gamma}{\gamma} r \mathbf{1}^{\mathbf{T}} \left[\left(\mathbf{V}\pi_\infty\right)\left(\mathbf{V}\pi_\infty\right)^{\mathbf{T}}\right]^{-\mathbf{1}} \mathbb{A}_{:,i} + \\
& + \frac{1-\gamma}{\gamma} \left[A_p \overline{\sigma}_{\mathbf{p}} \left(\mathbf{V}\pi_\infty\right)^{-1} \mathbb{A}_{:,i} - r C_{pi} \overline{\sigma}_{\mathbf{p}} \left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}} \mathbf{1} + \sum_{j=1}^{m} A_j \overline{\sigma}_{\mathbf{j},\pi} \left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}} \mathbb{A}_{:,i}\right] + \\
& - 2 \frac{1-\gamma}{\gamma} r B_i \overline{\sigma}_{\mathbf{i},\pi} \left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}} \mathbf{1} - \frac{1-\gamma}{\gamma} r \sum_{j \neq i} C_{ji} \overline{\sigma}_{\mathbf{j},\pi} \left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}} \mathbf{1}.
\end{aligned} \tag{10c}$$

Equation d (p_t^2 term):

$$\begin{aligned}
B_p' \;=\; & -\beta B_p + 2B_p^2 \sigma_{\mathbf{p}} \sigma_{\mathbf{p}}^{\mathbf{T}} + 2 \sum_{i=1}^{m} B_p C_{pi} \sigma_{\mathbf{p}} \sigma_{\mathbf{i},\pi}^{\mathbf{T}} + 2 \frac{1-\gamma}{\gamma} \sum_{i=1}^{m} B_p C_{pi} \overline{\sigma}_{\mathbf{p}} \overline{\sigma}_{\mathbf{i},\pi}^{\mathbf{T}} + \\
& + \frac{1}{2} \sum_{i,j=1}^{m} C_{pi} C_{pj} \sigma_{\mathbf{i},\pi} \sigma_{\mathbf{j},\pi}^{\mathbf{T}} + \frac{1-\gamma}{2\gamma} \sum_{i,j=1}^{m} C_{pi} C_{pj} \overline{\sigma}_{\mathbf{i},\pi} \overline{\sigma}_{\mathbf{j},\pi}^{\mathbf{T}} + \\
& + \frac{1-\gamma}{2\gamma} \mathbb{B}^{\mathbf{T}} \left[\left(\mathbf{V}\pi_\infty\right)\left(\mathbf{V}\pi_\infty\right)^{\mathbf{T}}\right]^{-\mathbf{1}} \mathbb{B} + 2 \frac{1-\gamma}{\gamma} B_p^2 \overline{\sigma}_{\mathbf{p}} \overline{\sigma}_{\mathbf{p}}^{\mathbf{T}} + \\
& + 2 \frac{1-\gamma}{\gamma} B_p \overline{\sigma}_{\mathbf{p}} \left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}} \mathbb{B} + \frac{1-\gamma}{\gamma} \sum_{i=1}^{m} C_{pi} \overline{\sigma}_{\mathbf{i},\pi} \left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}} \mathbb{B}.
\end{aligned} \tag{10d}$$

Equation e ($\pi_{i,t}^2$, $i \in R$, terms):

$$\begin{aligned}
B_i' &= -\beta B_i + C_{pi}\mu_{p,i} + \frac{1}{2}C_{pi}^2\sigma_{\mathbf{p}}\sigma_{\mathbf{p}}^{\mathbf{T}} + \frac{1-\gamma}{2\gamma}C_{pi}^2\overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{p}}^{\mathbf{T}} + 2C_{pi}B_i\sigma_{\mathbf{p}}\sigma_{\mathbf{i},\pi}^{\mathbf{T}} + \\
&+ 2\frac{1-\gamma}{\gamma}C_{pi}B_i\overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{i},\pi}^{\mathbf{T}} + \sum_{j\neq i} C_{pi}C_{ji}\sigma_{\mathbf{p}}\sigma_{\mathbf{j},\pi}^{\mathbf{T}} + \frac{1-\gamma}{\gamma}\sum_{j\neq i} C_{pi}C_{ji}\overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{j},\pi}^{\mathbf{T}} + \\
&+ 2\lambda_{ii}B_i + \sum_{j\neq i}\lambda_{ij}C_{ji} + 2B_i^2\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{i},\pi}^{\mathbf{T}} + 2\frac{1-\gamma}{\gamma}B_i^2\overline{\sigma}_{\mathbf{i},\pi}\overline{\sigma}_{\mathbf{i},\pi}^{\mathbf{T}} + \\
&+ 2B_i\sum_{j\neq i}C_{ji}\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{j},\pi}^{\mathbf{T}} + 2\frac{1-\gamma}{\gamma}B_i\sum_{j\neq i}C_{ji}\overline{\sigma}_{\mathbf{i},\pi}\overline{\sigma}_{\mathbf{j},\pi}^{\mathbf{T}} + \frac{1}{2}\sum_{j,k\neq i}C_{ji}C_{ki}\sigma_{\mathbf{j},\pi}\sigma_{\mathbf{k},\pi}^{\mathbf{T}} + \\
&+ \frac{1-\gamma}{2\gamma}\sum_{j,k\neq i}C_{ji}C_{ki}\overline{\sigma}_{\mathbf{j},\pi}\overline{\sigma}_{\mathbf{k},\pi}^{\mathbf{T}} + \frac{1-\gamma}{2\gamma}\mathbb{A}_{:,i}^{\mathbf{T}}\left[(\mathbf{V}\pi_\infty)(\mathbf{V}\pi_\infty)^{\mathbf{T}}\right]^{-\mathbf{1}}\mathbb{A}_{:,i} + \\
&+ \frac{1-\gamma}{\gamma}C_{pi}\overline{\sigma}_{\mathbf{p}}(\mathbf{V}\pi_\infty)^{-\mathbf{1}}\mathbb{A}_{:,i} + 2\frac{1-\gamma}{\gamma}B_i\overline{\sigma}_{\mathbf{i},\pi}(\mathbf{V}\pi_\infty)^{-\mathbf{1}}\mathbb{A}_{:,i} + \\
&+ \frac{1-\gamma}{\gamma}\sum_{j\neq i}C_{ji}\overline{\sigma}_{\mathbf{j},\pi}(\mathbf{V}\pi_\infty)^{-\mathbf{1}}\mathbb{A}_{:,i}.
\end{aligned} \tag{10e}$$

Equation f ($p_t\pi_{i,t}$, $i \in R$, terms):

$$\begin{aligned}
C_{pi}' &= -\beta C_{pi} + 2B_p\mu_{p,i} + 2B_pC_{pi}\sigma_{\mathbf{p}}\sigma_{\mathbf{p}}^{\mathbf{T}} + 2\frac{1-\gamma}{\gamma}B_pC_{pi}\overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{p}}^{\mathbf{T}} + 4B_pB_i\sigma_{\mathbf{p}}\sigma_{\mathbf{i},\pi}^{\mathbf{T}} + C_{pi}^2\sigma_{\mathbf{p}}\sigma_{\mathbf{i},\pi}^{\mathbf{T}} + \\
&+ \sum_{j\neq i}(2B_pC_{ji} + C_{pi}C_{pj})\sigma_{\mathbf{p}}\sigma_{\mathbf{j},\pi}^{\mathbf{T}} + 4\frac{1-\gamma}{\gamma}B_pB_i\overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{i},\pi}^{\mathbf{T}} + \frac{1-\gamma}{\gamma}C_{pi}^2\overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{i},\pi}^{\mathbf{T}} + \\
&+ \frac{1-\gamma}{\gamma}\sum_{j\neq i}(2B_pC_{ji} + C_{pi}C_{pj})\overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{j},\pi}^{\mathbf{T}} + \sum_{j=1}^{m}\lambda_{ij}C_{pj} + 2B_iC_{pi}\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{i},\pi}^{\mathbf{T}} + \\
&+ 2\frac{1-\gamma}{\gamma}B_iC_{pi}\overline{\sigma}_{\mathbf{i},\pi}\overline{\sigma}_{\mathbf{i},\pi}^{\mathbf{T}} + \sum_{j\neq i}(2B_iC_{pj} + C_{pi}C_{ji})\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{j},\pi}^{\mathbf{T}} + \\
&+ \frac{1-\gamma}{\gamma}\sum_{j\neq i}(2B_iC_{pj} + C_{pi}C_{ji})\overline{\sigma}_{\mathbf{i},\pi}\overline{\sigma}_{\mathbf{j},\pi}^{\mathbf{T}} + \frac{1}{2}\sum_{j,k\neq i}(C_{pj}C_{ki} + C_{pk}C_{ji})\sigma_{\mathbf{j},\pi}\sigma_{\mathbf{k},\pi}^{\mathbf{T}} + \\
&\frac{1-\gamma}{2\gamma}\sum_{j,k\neq i}(C_{pj}C_{ki} + C_{pk}C_{ji})\overline{\sigma}_{\mathbf{j},\pi}\overline{\sigma}_{\mathbf{k},\pi}^{\mathbf{T}} + \frac{1-\gamma}{\gamma}(\mathbb{A}_{:,i})^{\mathbf{T}}\left[(\mathbf{V}\pi_\infty)(\mathbf{V}\pi_\infty)^{\mathbf{T}}\right]^{-\mathbf{1}}\mathbb{B} + \\
&+ \frac{1-\gamma}{\gamma}\left[2B_p\overline{\sigma}_{\mathbf{p}}(\mathbf{V}\pi_\infty)^{-\mathbf{1}}\mathbb{A}_{:,i} + C_{pi}\overline{\sigma}_{\mathbf{p}}(\mathbf{V}\pi_\infty)^{-\mathbf{1}}\mathbb{B} + \sum_{j=1}^{m}C_{pj}\overline{\sigma}_{\mathbf{j},\pi}(\mathbf{V}\pi_\infty)^{-\mathbf{1}}\mathbb{A}_{:,i}\right] + \\
&+ 2\frac{1-\gamma}{\gamma}B_i\overline{\sigma}_{\mathbf{i},\pi}(\mathbf{V}\pi_\infty)^{-\mathbf{1}}\mathbb{B} + \frac{1-\gamma}{\gamma}\sum_{j\neq i}C_{ji}\overline{\sigma}_{\mathbf{j},\pi}(\mathbf{V}\pi_\infty)^{-\mathbf{1}}\mathbb{B}.
\end{aligned} \tag{10f}$$

Equation g ($\pi_{i,t}\pi_{j,t}$, $i < j \in R$, terms):

$$
\begin{aligned}
C'_{ij} &= -\beta C_{i,j} + C_{pi}\mu_{p,j} + C_{pj}\mu_{p,i} + C_{pi}C_{pj}\sigma_{\mathbf{p}}\sigma_{\mathbf{p}}^{\mathbf{T}} + \frac{1-\gamma}{\gamma}C_{pi}C_{pj}\sigma_{\mathbf{p}}\overline{\sigma}_{\mathbf{p}}^{\mathbf{T}} + \\
&+ \left(2B_iC_{pj} + C_{pi}C_{ij}\right)\sigma_{\mathbf{p}}\sigma_{\mathbf{i},\pi}^{\mathbf{T}} + \left(2B_jC_{pi} + C_{pj}C_{ij}\right)\sigma_{\mathbf{p}}\sigma_{\mathbf{j},\pi}^{\mathbf{T}} + \\
&+ \frac{1-\gamma}{\gamma}\left[\left(2B_iC_{pj} + C_{pi}C_{ij}\right)\overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{i},\pi}^{\mathbf{T}} + \left(2B_jC_{pi} + C_{pj}C_{ij}\right)\overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{j},\pi}^{\mathbf{T}}\right] + \\
&+ \sum_{k\neq i,j}\left(C_{pi}C_{kj} + C_{pj}C_{ki}\right)\sigma_{\mathbf{p}}\sigma_{\mathbf{k},\pi}^{\mathbf{T}} + \frac{1-\gamma}{\gamma}\sum_{k\neq i,j}\left(C_{pi}C_{kj} + C_{pj}C_{ki}\right)\overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{k},\pi}^{\mathbf{T}} + \\
&+ 2\lambda_{ij}B_j + 2\lambda_{ji}B_i + \lambda_{ii}C_{ij} + \lambda_{jj}C_{ij} + \sum_{k\neq i,j}\left(\lambda_{ik}C_{kj} + \lambda_{jk}C_{ki}\right) + \\
&+ 2B_iC_{ij}\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{i},\pi}^{\mathbf{T}} + 2B_jC_{ij}\sigma_{\mathbf{j},\pi}\sigma_{\mathbf{j},\pi}^{\mathbf{T}} + 2\frac{1-\gamma}{\gamma}\left[B_iC_{ij}\overline{\sigma}_{\mathbf{i},\pi}\overline{\sigma}_{\mathbf{i},\pi}^{\mathbf{T}} + B_jC_{ij}\overline{\sigma}_{\mathbf{j},\pi}\overline{\sigma}_{\mathbf{j},\pi}^{\mathbf{T}}\right] + \\
&+ \left(4B_iB_j + C_{ij}^2\right)\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{j},\pi}^{\mathbf{T}} + \frac{1-\gamma}{\gamma}\left(4B_iB_j + C_{ij}^2\right)\overline{\sigma}_{\mathbf{i},\pi}\overline{\sigma}_{\mathbf{j},\pi}^{\mathbf{T}} + \\
&+ \sum_{k\neq i,j}\left(2B_iC_{jk} + C_{ij}C_{ki}\right)\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{k},\pi}^{\mathbf{T}} + \sum_{k\neq i,j}\left(2B_jC_{ik} + C_{ji}C_{kj}\right)\sigma_{\mathbf{j},\pi}\sigma_{\mathbf{k},\pi}^{\mathbf{T}} + \\
&+ \frac{1-\gamma}{\gamma}\left[\sum_{k\neq i,j}\left(2B_iC_{jk} + C_{ij}C_{ki}\right)\overline{\sigma}_{\mathbf{i},\pi}\overline{\sigma}_{\mathbf{k},\pi}^{\mathbf{T}} + \sum_{k\neq i,j}\left(2B_jC_{ik} + C_{ji}C_{kj}\right)\overline{\sigma}_{\mathbf{j},\pi}\overline{\sigma}_{\mathbf{k},\pi}^{\mathbf{T}}\right] + \\
&+ \sum_{k<l\neq i,j}\left(C_{ki}C_{lj} + C_{kj}C_{li}\right)\sigma_{\mathbf{k},\pi}\sigma_{\mathbf{l},\pi}^{\mathbf{T}} + \sum_{k\neq i,j}C_{ik}C_{jk}\sigma_{\mathbf{k},\pi}\sigma_{\mathbf{k},\pi}^{\mathbf{T}} + \\
&+ \frac{1-\gamma}{\gamma}\left[\sum_{k<l\neq i,j}\left(C_{ki}C_{lj} + C_{kj}C_{li}\right)\overline{\sigma}_{\mathbf{k},\pi}\overline{\sigma}_{\mathbf{l},\pi}^{\mathbf{T}} + \sum_{k\neq i,j}C_{ik}C_{jk}\overline{\sigma}_{\mathbf{k},\pi}\overline{\sigma}_{\mathbf{k},\pi}^{\mathbf{T}}\right] + \\
&+ \frac{1-\gamma}{\gamma}\left(\mathbb{A}_{:,i}\right)^{\mathbf{T}}\left[\left(\mathbf{V}\pi_\infty\right)\left(\mathbf{V}\pi_\infty\right)^{\mathbf{T}}\right]^{-\mathbf{1}}\mathbb{A}_{:,j} + \\
&+ \frac{1-\gamma}{\gamma}C_{pi}\overline{\sigma}_{\mathbf{p}}\left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}}\mathbb{A}_{:,j} + \frac{1-\gamma}{\gamma}C_{pj}\overline{\sigma}_{\mathbf{p}}\left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}}\mathbb{A}_{:,i} + \\
&+ 2\frac{1-\gamma}{\gamma}B_i\overline{\sigma}_{\mathbf{i},\pi}\left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}}\mathbb{A}_{:,j} + \frac{1-\gamma}{\gamma}\sum_{k\neq i}C_{ik}\overline{\sigma}_{\mathbf{k},\pi}\left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}}\mathbb{A}_{:,j} + \\
&+ 2\frac{1-\gamma}{\gamma}B_j\overline{\sigma}_{\mathbf{j},\pi}\left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}}\mathbb{A}_{:,i} + \frac{1-\gamma}{\gamma}\sum_{k\neq j}C_{jk}\overline{\sigma}_{\mathbf{k},\pi}\left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}}\mathbb{A}_{:,i}.
\end{aligned}
\tag{10g}
$$

4 The Solution when $\psi \neq 1$

Under the general problem, the equation below is not linear due to its first term and therefore it is not analytically solvable.

$$
\begin{aligned}
0 \;=\; & \frac{1-\gamma}{\psi-1}\left(\beta^{\psi} e^{\frac{1-\psi}{1-\gamma} g} - \beta\psi\right) - A_0' - A_p' p_t - \sum_{i=1}^{m} A_i' \pi_{i,t} - B_p' p_t^2 - \sum_{i=1}^{m} B_i' \pi_{i,t}^2 - \sum_{i=1}^{m} C_{pi}' p_t \pi_{i,t} - \\
& - \sum_{i<j} C_{ij}' \pi_{i,t}\pi_{j,t} + (1-\gamma) r + \left(A_p + 2B_p p_t + \sum_{i=1}^{m} C_{pi}\pi_{i,t}\right) \mathbf{D_p}\boldsymbol{\pi_t} + \\
& + \frac{1}{2}\left[2B_p + \left(A_p + 2B_p p_t + \sum_{i=1}^{m} C_{pi}\pi_{i,t}\right)^2\right] \sigma_{\mathbf{p}}\sigma_{\mathbf{p}}^{\mathbf{T}} + \\
& + \sum_{i=1}^{m}\left[C_{pi} + \left(A_p + 2B_p p_t + \sum_{k=1}^{m} C_{pk}\pi_{k,t}\right)\left(A_i + 2B_i\pi_{i,t} + C_{pi}p_t + \sum_{k\neq i} C_{ik}\pi_{k,t}\right)\right] \sigma_{\mathbf{p}}\sigma_{\mathbf{i},\pi}^{\mathbf{T}} + \\
& + \frac{1-\gamma}{\gamma}\sum_{i=1}^{m}\left(A_p + 2B_p p_t + \sum_{k=1}^{m} C_{pk}\pi_{k,t}\right)\left(A_i + 2B_i\pi_{i,t} + C_{pi}p_t + \sum_{k\neq i} C_{ik}\pi_{k,t}\right) \overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{i},\pi}^{\mathbf{T}} + \\
& + \sum_{i,j=1}^{m} \lambda_{ji}\pi_{j,t}\left(A_i + 2B_i\pi_{i,t} + C_{pi}p_t + \sum_{k\neq i} C_{ik}\pi_{k,t}\right) + \sum_{i=1}^{m} B_i \sigma_{\mathbf{i},\pi}\sigma_{\mathbf{i},\pi}^{\mathbf{T}} + \sum_{i<j} C_{ij}\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{j},\pi}^{\mathbf{T}} + \\
& + \frac{1}{2}\sum_{i,j=1}^{m}\left(A_i + 2B_i\pi_{i,t} + C_{pi}p_t + \sum_{k\neq i} C_{ik}\pi_{k,t}\right)\left(A_j + 2B_j\pi_{j,t} + C_{pj}p_t + \sum_{k\neq j} C_{jk}\pi_{k,t}\right) \sigma_{\mathbf{i},\pi}\sigma_{\mathbf{j},\pi}^{\mathbf{T}} + \\
& + \frac{1}{2}\frac{1-\gamma}{\gamma}\sum_{i,j=1}^{m}\left(A_i + 2B_i\pi_{i,t} + C_{pi}p_t + \sum_{k\neq i} C_{ik}\pi_{k,t}\right)\left(A_j + 2B_j\pi_{j,t} + C_{pj}p_t + \sum_{k\neq j} C_{jk}\pi_{k,t}\right) \overline{\sigma}_{\mathbf{i},\pi}\overline{\sigma}_{\mathbf{j},\pi}^{\mathbf{T}} + \\
& + \frac{1}{2}\frac{1-\gamma}{\gamma}\left(\mathbb{A}\boldsymbol{\pi_t} + \mathbb{B}p_t - r\mathbf{1}\right)^{\mathbf{T}}\left[\left(\mathbf{V}\pi_\infty\right)\left(\mathbf{V}\pi_\infty\right)^{\mathbf{T}}\right]^{-\mathbf{1}}\left(\mathbb{A}\boldsymbol{\pi_t} + \mathbb{B}p_t - r\mathbf{1}\right) + \\
& + \frac{1}{2}\frac{1-\gamma}{\gamma}\left(A_p + 2B_p p_t + \sum_{i=1}^{m} C_{pi}\pi_{i,t}\right)^2 \overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{p}}^{\mathbf{T}} + \\
& + \frac{1-\gamma}{\gamma}\left(A_p + 2B_p p_t + \sum_{i=1}^{m} C_{pi}\pi_{i,t}\right)\overline{\sigma}_{\mathbf{p}}\left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}}\left(\mathbb{A}\boldsymbol{\pi_t} + \mathbb{B}p_t - r\mathbf{1}\right) + \\
& + \frac{1-\gamma}{\gamma}\sum_{i=1}^{m}\left(A_i + 2B_i\pi_{i,t} + C_{pi}p_t + \sum_{k\neq i} C_{ik}\pi_{k,t}\right)\overline{\sigma}_{\mathbf{i},\pi}\left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}}\left(\mathbb{A}\boldsymbol{\pi_t} + \mathbb{B}p_t - r\mathbf{1}\right).
\end{aligned}
\tag{11}
$$

However, this term turns out to be (proportional to) the consumption to wealth ratio, what can be shown by substituting the value function in the first-order condition for consumption:

$$
\frac{C_t}{W_t} = \beta^{\psi} e^{\frac{1-\psi}{1-\gamma} g(p_t,\boldsymbol{\pi_t},\tau)}. \tag{12}
$$

We will apply exactly the same methodology applied in Campani et al. (2014), based on log-linear techniques. The discussion there remains the same here: the point around which the approximation will be made can be chosen by the investor according to current conditions, beliefs and investment horizon. The state variable space is given here by the predictor and π_t. In this study, to be consistent with the volatility approximation made, we will approximate around the (log of) consumption to wealth ratio value under the ergodic probabilities. Regarding the predictor, we choose its historical mean (although any value could be chosen), denoted by $\overline{p}$:

$$\begin{aligned} \beta^{\psi} e^{\frac{1-\psi}{1-\gamma} g(p_t,\pi_t,\tau)} &= \exp\left(\ln \frac{C_t}{W_t}\right) \approx \\ \approx \exp\left(\ln \frac{C_t}{W_t}\right)_{p_t=\overline{p},\pi_t=\pi_\infty} + \exp\left(\ln \frac{C_t}{W_t}\right)_{p_t=\overline{p},\pi_t=\pi_\infty} &\left[\ln \frac{C_t}{W_t} - \left(\ln \frac{C_t}{W_t}\right)_{p_t=\overline{p},\pi_t=\pi_\infty}\right] = \\ = \beta^{\psi} e^{\frac{1-\psi}{1-\gamma} g(\overline{p},\pi_\infty,\tau)} * &\left\{1 + \left[\ln \frac{C_t}{W_t} - \left(\ln \frac{C_t}{W_t}\right)_{p_t=\overline{p},\pi_t=\pi_\infty}\right]\right\}. \end{aligned} \tag{13}$$

The approximation 13 renders equation 11 solvable, with a solution given by:

$$\begin{aligned} g(p_t,\pi_t,\tau) = {} & A_0(\tau) + A_p(\tau)\, p_t + \sum_{i=1}^{m} A_i(\tau)\, \pi_{i,t} + B_p(\tau)\, p_t^2 + \\ & + \sum_{i=1}^{m} B_i(\tau)\, \pi_{i,t}^2 + \sum_{i=1}^{m} C_{pi}(\tau)\, p_t \pi_{i,t} + \sum_{i<j} C_{ij}(\tau)\, \pi_{i,t}\pi_{j,t}. \end{aligned} \tag{14}$$

The partial differential equation to be solved is then the previous one with the log-linearization:

$$
\begin{aligned}
0 \; &= \beta^{\psi} e^{\frac{1-\psi}{1-\gamma}\left(A_0(\tau)+A_p(\tau)\overline{p}+\sum_{i=1}^{m} A_i(\tau)\pi_{i,\infty}+B_p(\tau)\overline{p}^2+\sum_{i=1}^{m} B_i(\tau)\pi_{i,\infty}^2+\sum_{i=1}^{m} C_{pi}(\tau)\overline{p}\pi_{i,\infty}+\sum_{i<j} C_{ij}(\tau)\pi_{i,\infty}\pi_{j,\infty}\right)} * \\
&* \left[\begin{array}{l}\frac{1-\gamma}{\psi-1} - A_p\left(p_t-\overline{p}\right) - \sum_{i=1}^{m} A_i\left(\pi_{i,t}-\pi_{i,\infty}\right) - B_p\left(p_t^2-\overline{p}^2\right) - \sum_{i=1}^{m} B_i\left(\pi_{i,t}^2-\pi_{i,\infty}^2\right) - \\ - \sum_{i=1}^{m} C_{pi}\left(p_t\pi_{i,t}-\overline{p}\pi_{i,\infty}\right) - \sum_{i<j} C_{ij}\left(\pi_{i,t}\pi_{j,t}-\pi_{i,\infty}\pi_{j,\infty}\right)\end{array}\right] - \\
&- \frac{1-\gamma}{\psi-1}\beta\psi - A_0' - A_p' p_t - \sum_{i=1}^{m} A_i'\pi_{i,t} - B_p' p_t^2 - \sum_{i=1}^{m} B_i'\pi_{i,t}^2 - \sum_{i=1}^{m} C_{pi}' p_t \pi_{i,t} - \\
&- \sum_{i<j} C_{ij}'\pi_{i,t}\pi_{j,t} + (1-\gamma)\, r + \left(A_p + 2B_p p_t + \sum_{i=1}^{m} C_{pi}\pi_{i,t}\right)\mathbf{D_p}\pi_{\mathbf{t}} + \\
&+ \frac{1}{2}\left[2B_p + \left(A_p + 2B_p p_t + \sum_{i=1}^{m} C_{pi}\pi_{i,t}\right)^2\right]\sigma_{\mathbf{p}}\sigma_{\mathbf{p}}^{\mathbf{T}} + \\
&+ \sum_{i=1}^{m}\left[C_{pi} + \left(A_p + 2B_p p_t + \sum_{k=1}^{m} C_{pk}\pi_{k,t}\right)\left(A_i + 2B_i\pi_{i,t} + C_{pi}p_t + \sum_{k\neq i} C_{ik}\pi_{k,t}\right)\right]\sigma_{\mathbf{p}}\sigma_{\mathbf{i},\pi}^{\mathbf{T}} + \\
&+ \frac{1-\gamma}{\gamma}\sum_{i=1}^{m}\left(A_p + 2B_p p_t + \sum_{k=1}^{m} C_{pk}\pi_{k,t}\right)\left(A_i + 2B_i\pi_{i,t} + C_{pi}p_t + \sum_{k\neq i} C_{ik}\pi_{k,t}\right)\overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{i},\pi}^{\mathbf{T}} + \\
&+ \sum_{i,j=1}^{m}\lambda_{ji}\pi_{j,t}\left(A_i + 2B_i\pi_{i,t} + C_{pi}p_t + \sum_{k\neq i} C_{ik}\pi_{k,t}\right) + \sum_{i=1}^{m} B_i\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{i},\pi}^{\mathbf{T}} + \sum_{i<j} C_{ij}\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{j},\pi}^{\mathbf{T}} + \\
&+ \frac{1}{2}\sum_{i,j=1}^{m}\left(A_i + 2B_i\pi_{i,t} + C_{pi}p_t + \sum_{k\neq i} C_{ik}\pi_{k,t}\right)\left(A_j + 2B_j\pi_{j,t} + C_{pj}p_t + \sum_{k\neq j} C_{jk}\pi_{k,t}\right)\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{j},\pi}^{\mathbf{T}} + \\
&+ \frac{1}{2}\sum_{i,j=1}^{m}\frac{1-\gamma}{\gamma}\left(A_i + 2B_i\pi_{i,t} + C_{pi}p_t + \sum_{k\neq i} C_{ik}\pi_{k,t}\right)\left(A_j + 2B_j\pi_{j,t} + C_{pj}p_t + \sum_{k\neq j} C_{jk}\pi_{k,t}\right)\overline{\sigma}_{\mathbf{i},\pi}\overline{\sigma}_{\mathbf{j},\pi}^{\mathbf{T}} + \\
&+ \frac{1}{2}\frac{1-\gamma}{\gamma}\left(\mathbb{A}\pi_{\mathbf{t}} + \mathbb{B}p_t - r\mathbf{1}\right)^{\mathbf{T}}\left[\left(\mathbf{V}\pi_\infty\right)\left(\mathbf{V}\pi_\infty\right)^{\mathbf{T}}\right]^{-\mathbf{1}}\left(\mathbb{A}\pi_{\mathbf{t}} + \mathbb{B}p_t - r\mathbf{1}\right) + \\
&+ \frac{1}{2}\frac{1-\gamma}{\gamma}\left(A_p + 2B_p p_t + \sum_{i=1}^{m} C_{pi}\pi_{i,t}\right)^2\overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{p}}^{\mathbf{T}} + \\
&+ \frac{1-\gamma}{\gamma}\left(A_p + 2B_p p_t + \sum_{i=1}^{m} C_{pi}\pi_{i,t}\right)\overline{\sigma}_{\mathbf{p}}\left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}}\left(\mathbb{A}\pi_{\mathbf{t}} + \mathbb{B}p_t - r\mathbf{1}\right) + \\
&+ \frac{1-\gamma}{\gamma}\sum_{i=1}^{m}\left(A_i + 2B_i\pi_{i,t} + C_{pi}p_t + \sum_{k\neq i} C_{ik}\pi_{k,t}\right)\overline{\sigma}_{\mathbf{i},\pi}\left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}}\left(\mathbb{A}\pi_{\mathbf{t}} + \mathbb{B}p_t - r\mathbf{1}\right).
\end{aligned}
\tag{15}
$$

As before, we match the coefficients and solve a system to find the time-varying coefficients of function $g\left(p_t, \pi_{\mathbf{t}}, \tau\right)$. The system follows.

Equation a (independent term):

$$\begin{aligned}
A_0' &= \beta^{\psi} e^{\frac{1-\psi}{1-\gamma}\left(A_0 + A_p\overline{p} + \sum_{i=1}^{m} A_i \pi_{i,\infty} + B_p \overline{p}^2 + \sum_{i=1}^{m} B_i \pi_{i,\infty}^2 + \sum_{i=1}^{m} C_{pi}\overline{p}\pi_{i,\infty} + \sum_{i<j} C_{ij}\pi_{i,\infty}\pi_{j,\infty}\right)} * \\
&* \left(\frac{1-\gamma}{\psi-1} + A_p\overline{p} + \sum_{i=1}^{m} A_i\pi_{i,\infty} + B_p\overline{p}^2 + \sum_{i=1}^{m} B_i\pi_{i,\infty}^2 + \sum_{i=1}^{m} C_{pi}\overline{p}\pi_{i,\infty} + \sum_{i<j} C_{ij}\pi_{i,\infty}\pi_{j,\infty}\right) + \\
&+ (1-\gamma) r - \frac{1-\gamma}{\psi-1}\beta\psi + \left(B_p + \frac{1}{2}A_p^2\right)\sigma_{\mathbf{p}}\sigma_{\mathbf{p}}^{\mathbf{T}} + \frac{1-\gamma}{2\gamma}A_p^2\overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{p}}^{\mathbf{T}} + \sum_{i=1}^{m}\left(C_{pi} + A_pA_i\right)\sigma_{\mathbf{p}}\sigma_{\mathbf{i},\pi}^{\mathbf{T}} + \\
&+ \frac{1-\gamma}{\gamma}A_p\sum_{i=1}^{m} A_i\overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{i},\pi}^{\mathbf{T}} + \sum_{i=1}^{m} B_i\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{i},\pi}^{\mathbf{T}} + \sum_{i<j} C_{ij}\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{j},\pi}^{\mathbf{T}} + \frac{1}{2}\sum_{i,j=1}^{m} A_iA_j\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{j},\pi}^{\mathbf{T}} + \\
&+ \frac{1-\gamma}{2\gamma}\sum_{i,j=1}^{m} A_iA_j\overline{\sigma}_{\mathbf{i},\pi}\overline{\sigma}_{\mathbf{j},\pi}^{\mathbf{T}} + \frac{1-\gamma}{2\gamma}r^2\mathbf{1}^{\mathbf{T}}\left[\left(\mathbf{V}\pi_\infty\right)\left(\mathbf{V}\pi_\infty\right)^{\mathbf{T}}\right]^{-1}\mathbf{1} - \frac{1-\gamma}{\gamma}rA_p\overline{\sigma}_{\mathbf{p}}\left(\mathbf{V}\pi_\infty\right)^{-1}\mathbf{1} - \\
&- \frac{1-\gamma}{\gamma}r\sum_{i=1}^{m} A_i\overline{\sigma}_{\mathbf{i},\pi}\left(\mathbf{V}\pi_\infty\right)^{-1}\mathbf{1}.
\end{aligned} \tag{16a}$$

Equation b (p_t term):

$$\begin{aligned}
A_p' &= -A_p\beta^{\psi} e^{\frac{1-\psi}{1-\gamma}\left(A_0 + A_p\overline{p} + \sum_{i=1}^{m} A_i \pi_{i,\infty} + B_p \overline{p}^2 + \sum_{i=1}^{m} B_i \pi_{i,\infty}^2 + \sum_{i=1}^{m} C_{pi}\overline{p}\pi_{i,\infty} + \sum_{i<j} C_{ij}\pi_{i,\infty}\pi_{j,\infty}\right)} + \\
&+ 2A_pB_p\sigma_{\mathbf{p}}\sigma_{\mathbf{p}}^{\mathbf{T}} + 2\frac{1-\gamma}{\gamma}A_pB_p\overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{p}}^{\mathbf{T}} + \sum_{i=1}^{m}\left(A_pC_{pi} + 2A_iB_p\right)\sigma_{\mathbf{p}}\sigma_{\mathbf{i},\pi}^{\mathbf{T}} + \\
&+ \frac{1-\gamma}{\gamma}\sum_{i=1}^{m}\left(A_pC_{pi} + 2A_iB_p\right)\overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{i},\pi}^{\mathbf{T}} + \frac{1}{2}\sum_{i,j=1}^{m}\left(A_iC_{pj} + A_jC_{pi}\right)\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{j},\pi}^{\mathbf{T}} + \\
&+ \frac{1-\gamma}{2\gamma}\sum_{i,j=1}^{m}\left(A_iC_{pj} + A_jC_{pi}\right)\overline{\sigma}_{\mathbf{i},\pi}\overline{\sigma}_{\mathbf{j},\pi}^{\mathbf{T}} - \frac{1-\gamma}{\gamma}r\mathbb{B}^{\mathbf{T}}\left[\left(\mathbf{V}\pi_\infty\right)\left(\mathbf{V}\pi_\infty\right)^{\mathbf{T}}\right]^{-1}\mathbf{1} + \\
&+ \frac{1-\gamma}{\gamma}A_p\overline{\sigma}_{\mathbf{p}}\left(\mathbf{V}\pi_\infty\right)^{-1}\mathbb{B} - 2\frac{1-\gamma}{\gamma}rB_p\overline{\sigma}_{\mathbf{p}}\left(\mathbf{V}\pi_\infty\right)^{-1}\mathbf{1} + \\
&+ \frac{1-\gamma}{\gamma}\sum_{i=1}^{m} A_i\overline{\sigma}_{\mathbf{i},\pi}\left(\mathbf{V}\pi_\infty\right)^{-1}\mathbb{B} - \frac{1-\gamma}{\gamma}r\sum_{i=1}^{m} C_{pi}\overline{\sigma}_{\mathbf{i},\pi}\left(\mathbf{V}\pi_\infty\right)^{-1}\mathbf{1}.
\end{aligned} \tag{16b}$$

Equation c ($\pi_{i,t}$, $i \in R$, terms):

$$\begin{aligned}
A_i' \;=\; & -A_i\beta^{\psi} e^{\frac{1-\psi}{1-\gamma}\left(A_0 + A_p\overline{p} + \sum_{i=1}^{m} A_i\pi_{i,\infty} + B_p\overline{p}^2 + \sum_{i=1}^{m} B_i\pi_{i,\infty}^2 + \sum_{i=1}^{m} C_{pi}\overline{p}\pi_{i,\infty} + \sum_{i<j} C_{ij}\pi_{i,\infty}\pi_{j,\infty}\right)} + \\
& + A_p\mu_{p,i} + A_pC_{pi}\sigma_{\mathbf{p}}\sigma_{\mathbf{p}}^{\mathbf{T}} + \frac{1-\gamma}{\gamma}A_pC_{pi}\overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{p}}^{\mathbf{T}} + \left(2A_pB_i + A_iC_{pi}\right)\sigma_{\mathbf{p}}\sigma_{\mathbf{i},\pi}^{\mathbf{T}} + \\
& + \sum_{j\neq i}\left(A_pC_{ji} + A_jC_{pi}\right)\sigma_{\mathbf{p}}\sigma_{\mathbf{j},\pi}^{\mathbf{T}} + \frac{1-\gamma}{\gamma}\left(2A_pB_i + A_iC_{pi}\right)\overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{i},\pi}^{\mathbf{T}} + \\
& + \frac{1-\gamma}{\gamma}\sum_{j\neq i}\left(A_pC_{ji} + A_jC_{pi}\right)\overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{j},\pi}^{\mathbf{T}} + \sum_{j=1}^{m}\lambda_{ij}A_j + 2A_iB_i\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{i},\pi}^{\mathbf{T}} + \\
& + \sum_{j\neq i}\left(A_iC_{ji} + 2A_jB_i\right)\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{j},\pi}^{\mathbf{T}} + \frac{1}{2}\sum_{j,k\neq i}\left(A_jC_{ki} + A_kC_{ji}\right)\sigma_{\mathbf{j},\pi}\sigma_{\mathbf{k},\pi}^{\mathbf{T}} + \\
& + 2\frac{1-\gamma}{\gamma}A_iB_i\overline{\sigma}_{\mathbf{i},\pi}\overline{\sigma}_{\mathbf{i},\pi}^{\mathbf{T}} + \frac{1-\gamma}{\gamma}\sum_{j\neq i}\left(A_iC_{ji} + 2A_jB_i\right)\overline{\sigma}_{\mathbf{i},\pi}\overline{\sigma}_{\mathbf{j},\pi}^{\mathbf{T}} + \\
& + \frac{1-\gamma}{2\gamma}\sum_{j,k\neq i}\left(A_jC_{ki} + A_kC_{ji}\right)\overline{\sigma}_{\mathbf{j},\pi}\overline{\sigma}_{\mathbf{k},\pi}^{\mathbf{T}} - \frac{1-\gamma}{\gamma}r\mathbf{1}^{\mathbf{T}}\left[\left(\mathbf{V}\pi_{\infty}\right)\left(\mathbf{V}\pi_{\infty}\right)^{\mathbf{T}}\right]^{-\mathbf{1}}\mathbb{A}_{:,i} + \\
& + \frac{1-\gamma}{\gamma}\left[A_p\overline{\sigma}_{\mathbf{p}}\left(\mathbf{V}\pi_{\infty}\right)^{-\mathbf{1}}\mathbb{A}_{:,i} - rC_{pi}\overline{\sigma}_{\mathbf{p}}\left(\mathbf{V}\pi_{\infty}\right)^{-\mathbf{1}}\mathbf{1} + \sum_{j=1}^{m}A_j\overline{\sigma}_{\mathbf{j},\pi}\left(\mathbf{V}\pi_{\infty}\right)^{-\mathbf{1}}\mathbb{A}_{:,i}\right] + \\
& - 2\frac{1-\gamma}{\gamma}rB_i\overline{\sigma}_{\mathbf{i},\pi}\left(\mathbf{V}\pi_{\infty}\right)^{-\mathbf{1}}\mathbf{1} - \frac{1-\gamma}{\gamma}r\sum_{j\neq i}C_{ji}\overline{\sigma}_{\mathbf{j},\pi}\left(\mathbf{V}\pi_{\infty}\right)^{-\mathbf{1}}\mathbf{1}.
\end{aligned} \tag{16c}$$

Equation d (p_t^2 term):

$$\begin{aligned}
B_p' \;=\; & -B_p\beta^{\psi} e^{\frac{1-\psi}{1-\gamma}\left(A_0 + A_p\overline{p} + \sum_{i=1}^{m} A_i\pi_{i,\infty} + B_p\overline{p}^2 + \sum_{i=1}^{m} B_i\pi_{i,\infty}^2 + \sum_{i=1}^{m} C_{pi}\overline{p}\pi_{i,\infty} + \sum_{i<j} C_{ij}\pi_{i,\infty}\pi_{j,\infty}\right)} + \\
& + 2B_p^2\sigma_{\mathbf{p}}\sigma_{\mathbf{p}}^{\mathbf{T}} + 2\sum_{i=1}^{m}B_pC_{pi}\sigma_{\mathbf{p}}\sigma_{\mathbf{i},\pi}^{\mathbf{T}} + 2\frac{1-\gamma}{\gamma}\sum_{i=1}^{m}B_pC_{pi}\overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{i},\pi}^{\mathbf{T}} + \\
& + \frac{1}{2}\sum_{i,j=1}^{m}C_{pi}C_{pj}\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{j},\pi}^{\mathbf{T}} + \frac{1-\gamma}{2\gamma}\sum_{i,j=1}^{m}C_{pi}C_{pj}\overline{\sigma}_{\mathbf{i},\pi}\overline{\sigma}_{\mathbf{j},\pi}^{\mathbf{T}} + \\
& + \frac{1-\gamma}{2\gamma}\mathbb{B}^{\mathbf{T}}\left[\left(\mathbf{V}\pi_{\infty}\right)\left(\mathbf{V}\pi_{\infty}\right)^{\mathbf{T}}\right]^{-\mathbf{1}}\mathbb{B} + 2\frac{1-\gamma}{\gamma}B_p^2\overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{p}}^{\mathbf{T}} + \\
& + 2\frac{1-\gamma}{\gamma}B_p\overline{\sigma}_{\mathbf{p}}\left(\mathbf{V}\pi_{\infty}\right)^{-\mathbf{1}}\mathbb{B} + \frac{1-\gamma}{\gamma}\sum_{i=1}^{m}C_{pi}\overline{\sigma}_{\mathbf{i},\pi}\left(\mathbf{V}\pi_{\infty}\right)^{-\mathbf{1}}\mathbb{B}.
\end{aligned} \tag{16d}$$

Equation e ($\pi_{i,t}^2$, $i \in R$, terms):

$$
\begin{aligned}
B_i' \; = \; & -B_i\beta^{\psi} e^{\frac{1-\psi}{1-\gamma}\left(A_0+A_p\overline{p}+\sum_{i=1}^{m} A_i\pi_{i,\infty}+B_p\overline{p}^2+\sum_{i=1}^{m} B_i\pi_{i,\infty}^2+\sum_{i=1}^{m} C_{pi}\overline{p}\pi_{i,\infty}+\sum_{i<j} C_{ij}\pi_{i,\infty}\pi_{j,\infty}\right)}+ \\
& + C_{pi}\mu_{p,i} + \frac{1}{2}C_{pi}^2\sigma_{\mathbf{p}}\sigma_{\mathbf{p}}^{\mathbf{T}} + \frac{1-\gamma}{2\gamma}C_{pi}^2\overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{p}}^{\mathbf{T}} + 2C_{pi}B_i\sigma_{\mathbf{p}}\sigma_{\mathbf{i},\pi}^{\mathbf{T}}+ \\
& + 2\frac{1-\gamma}{\gamma}C_{pi}B_i\overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{i},\pi}^{\mathbf{T}} + \sum_{j\neq i} C_{pi}C_{ji}\sigma_{\mathbf{p}}\sigma_{\mathbf{j},\pi}^{\mathbf{T}} + \frac{1-\gamma}{\gamma}\sum_{j\neq i} C_{pi}C_{ji}\overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{j},\pi}^{\mathbf{T}}+ \\
& + 2\lambda_{ii}B_i + \sum_{j\neq i}\lambda_{ij}C_{ji} + 2B_i^2\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{i},\pi}^{\mathbf{T}}+2\frac{1-\gamma}{\gamma}B_i^2\overline{\sigma}_{\mathbf{i},\pi}\overline{\sigma}_{\mathbf{i},\pi}^{\mathbf{T}}+ \\
& + 2B_i\sum_{j\neq i} C_{ji}\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{j},\pi}^{\mathbf{T}} + 2\frac{1-\gamma}{\gamma}B_i\sum_{j\neq i} C_{ji}\overline{\sigma}_{\mathbf{i},\pi}\overline{\sigma}_{\mathbf{j},\pi}^{\mathbf{T}} + \frac{1}{2}\sum_{j,k\neq i} C_{ji}C_{ki}\sigma_{\mathbf{j},\pi}\sigma_{\mathbf{k},\pi}^{\mathbf{T}}+ \\
& + \frac{1-\gamma}{2\gamma}\sum_{j,k\neq i} C_{ji}C_{ki}\overline{\sigma}_{\mathbf{j},\pi}\overline{\sigma}_{\mathbf{k},\pi}^{\mathbf{T}}+\frac{1-\gamma}{2\gamma}\mathbb{A}_{:,i}^{\mathbf{T}}\left[\left(\mathbf{V}\pi_{\infty}\right)\left(\mathbf{V}\pi_{\infty}\right)^{\mathbf{T}}\right]^{-\mathbf{1}}\mathbb{A}_{:,i}+ \\
& + \frac{1-\gamma}{\gamma}C_{pi}\overline{\sigma}_{\mathbf{p}}\left(\mathbf{V}\pi_{\infty}\right)^{-\mathbf{1}}\mathbb{A}_{:,i} + 2\frac{1-\gamma}{\gamma}B_i\overline{\sigma}_{\mathbf{i},\pi}\left(\mathbf{V}\pi_{\infty}\right)^{-\mathbf{1}}\mathbb{A}_{:,i}+ \\
& + \frac{1-\gamma}{\gamma}\sum_{j\neq i} C_{ji}\overline{\sigma}_{\mathbf{j},\pi}\left(\mathbf{V}\pi_{\infty}\right)^{-\mathbf{1}}\mathbb{A}_{:,i}.
\end{aligned}
\tag{16e}
$$

Equation f ($p_t\pi_{i,t}$, $i \in R$, terms):

$$
\begin{aligned}
C_{pi}' \; = \; & -C_{pi}\beta^{\psi} e^{\frac{1-\psi}{1-\gamma}\left(A_0+A_p\overline{p}+\sum_{i=1}^{m} A_i\pi_{i,\infty}+B_p\overline{p}^2+\sum_{i=1}^{m} B_i\pi_{i,\infty}^2+\sum_{i=1}^{m} C_{pi}\overline{p}\pi_{i,\infty}+\sum_{i<j} C_{ij}\pi_{i,\infty}\pi_{j,\infty}\right)}+ \\
& + 2B_p\mu_{p,i} + 2B_pC_{pi}\sigma_{\mathbf{p}}\sigma_{\mathbf{p}}^{\mathbf{T}} + 2\frac{1-\gamma}{\gamma}B_pC_{pi}\overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{p}}^{\mathbf{T}} + 4B_pB_i\sigma_{\mathbf{p}}\sigma_{\mathbf{i},\pi}^{\mathbf{T}} + C_{pi}^2\sigma_{\mathbf{p}}\sigma_{\mathbf{i},\pi}^{\mathbf{T}}+ \\
& + \sum_{j\neq i}\left(2B_pC_{ji} + C_{pi}C_{pj}\right)\sigma_{\mathbf{p}}\sigma_{\mathbf{j},\pi}^{\mathbf{T}} + 4\frac{1-\gamma}{\gamma}B_pB_i\overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{i},\pi}^{\mathbf{T}}+\frac{1-\gamma}{\gamma}C_{pi}^2\overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{i},\pi}^{\mathbf{T}}+ \\
& + \frac{1-\gamma}{\gamma}\sum_{j\neq i}\left(2B_pC_{ji} + C_{pi}C_{pj}\right)\overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{j},\pi}^{\mathbf{T}}+\sum_{j=1}^{m}\lambda_{ij}C_{pj} + 2B_iC_{pi}\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{i},\pi}^{\mathbf{T}}+ \\
& + 2\frac{1-\gamma}{\gamma}B_iC_{pi}\overline{\sigma}_{\mathbf{i},\pi}\overline{\sigma}_{\mathbf{i},\pi}^{\mathbf{T}} + \sum_{j\neq i}\left(2B_iC_{pj} + C_{pi}C_{ji}\right)\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{j},\pi}^{\mathbf{T}}+ \\
& + \frac{1-\gamma}{\gamma}\sum_{j\neq i}\left(2B_iC_{pj} + C_{pi}C_{ji}\right)\overline{\sigma}_{\mathbf{i},\pi}\overline{\sigma}_{\mathbf{j},\pi}^{\mathbf{T}} + \frac{1}{2}\sum_{j,k\neq i}\left(C_{pj}C_{ki} + C_{pk}C_{ji}\right)\sigma_{\mathbf{j},\pi}\sigma_{\mathbf{k},\pi}^{\mathbf{T}}+ \\
& \frac{1-\gamma}{2\gamma}\sum_{j,k\neq i}\left(C_{pj}C_{ki} + C_{pk}C_{ji}\right)\overline{\sigma}_{\mathbf{j},\pi}\overline{\sigma}_{\mathbf{k},\pi}^{\mathbf{T}} + \frac{1-\gamma}{\gamma}\left(\mathbb{A}_{:,i}\right)^{\mathbf{T}}\left[\left(\mathbf{V}\pi_{\infty}\right)\left(\mathbf{V}\pi_{\infty}\right)^{\mathbf{T}}\right]^{-\mathbf{1}}\mathbb{B}+ \\
& + \frac{1-\gamma}{\gamma}\left[2B_p\overline{\sigma}_{\mathbf{p}}\left(\mathbf{V}\pi_{\infty}\right)^{-\mathbf{1}}\mathbb{A}_{:,i} + C_{pi}\overline{\sigma}_{\mathbf{p}}\left(\mathbf{V}\pi_{\infty}\right)^{-\mathbf{1}}\mathbb{B} + \sum_{j=1}^{m} C_{pj}\overline{\sigma}_{\mathbf{j},\pi}\left(\mathbf{V}\pi_{\infty}\right)^{-\mathbf{1}}\mathbb{A}_{:,i}\right] + \\
& + 2\frac{1-\gamma}{\gamma}B_i\overline{\sigma}_{\mathbf{i},\pi}\left(\mathbf{V}\pi_{\infty}\right)^{-\mathbf{1}}\mathbb{B} + \frac{1-\gamma}{\gamma}\sum_{j\neq i} C_{ji}\overline{\sigma}_{\mathbf{j},\pi}\left(\mathbf{V}\pi_{\infty}\right)^{-\mathbf{1}}\mathbb{B}.
\end{aligned}
\tag{16f}
$$

Equation g ($\pi_{i,t}\pi_{j,t}$, $i<j\in R$, terms):

$$
\begin{aligned}
C'_{ij} \;=\; & -C_{ij}\beta^{\psi}e^{\frac{1-\psi}{1-\gamma}\left(A_0+A_p\overline{p}+\sum\limits_{i=1}^{m}A_i\pi_{i,\infty}+B_p\overline{p}^2+\sum\limits_{i=1}^{m}B_i\pi_{i,\infty}^2+\sum\limits_{i=1}^{m}C_{pi}\overline{p}\pi_{i,\infty}+\sum\limits_{i<j}C_{ij}\pi_{i,\infty}\pi_{j,\infty}\right)}+\\
& + C_{pi}\mu_{p,j}+C_{pj}\mu_{p,i}+C_{pi}C_{pj}\sigma_{\mathbf{p}}\sigma_{\mathbf{p}}^{\mathbf{T}}+\frac{1-\gamma}{\gamma}C_{pi}C_{pj}\sigma_{\mathbf{p}}\overline{\sigma}_{\mathbf{p}}^{\mathbf{T}}+\\
& + \left(2B_iC_{pj}+C_{pi}C_{ij}\right)\sigma_{\mathbf{p}}\sigma_{\mathbf{i},\pi}^{\mathbf{T}}+\left(2B_jC_{pi}+C_{pj}C_{ij}\right)\sigma_{\mathbf{p}}\sigma_{\mathbf{j},\pi}^{\mathbf{T}}+\\
& + \frac{1-\gamma}{\gamma}\left[\left(2B_iC_{pj}+C_{pi}C_{ij}\right)\overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{i},\pi}^{\mathbf{T}}+\left(2B_jC_{pi}+C_{pj}C_{ij}\right)\overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{j},\pi}^{\mathbf{T}}\right]+\\
& + \sum_{k\neq i,j}\left(C_{pi}C_{kj}+C_{pj}C_{ki}\right)\sigma_{\mathbf{p}}\sigma_{\mathbf{k},\pi}^{\mathbf{T}}+\frac{1-\gamma}{\gamma}\sum_{k\neq i,j}\left(C_{pi}C_{kj}+C_{pj}C_{ki}\right)\overline{\sigma}_{\mathbf{p}}\overline{\sigma}_{\mathbf{k},\pi}^{\mathbf{T}}+\\
& + 2\lambda_{ij}B_j+2\lambda_{ji}B_i+\lambda_{ii}C_{ij}+\lambda_{jj}C_{ij}+\sum_{k\neq i,j}\left(\lambda_{ik}C_{kj}+\lambda_{jk}C_{ki}\right)+\\
& + 2B_iC_{ij}\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{i},\pi}^{\mathbf{T}}+2B_jC_{ij}\sigma_{\mathbf{j},\pi}\sigma_{\mathbf{j},\pi}^{\mathbf{T}}+2\frac{1-\gamma}{\gamma}\left[B_iC_{ij}\overline{\sigma}_{\mathbf{i},\pi}\overline{\sigma}_{\mathbf{i},\pi}^{\mathbf{T}}+B_jC_{ij}\overline{\sigma}_{\mathbf{j},\pi}\overline{\sigma}_{\mathbf{j},\pi}^{\mathbf{T}}\right]+\\
& + \left(4B_iB_j+C_{ij}^2\right)\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{j},\pi}^{\mathbf{T}}+\frac{1-\gamma}{\gamma}\left(4B_iB_j+C_{ij}^2\right)\overline{\sigma}_{\mathbf{i},\pi}\overline{\sigma}_{\mathbf{j},\pi}^{\mathbf{T}}+\\
& + \sum_{k\neq i,j}\left(2B_iC_{jk}+C_{ij}C_{ki}\right)\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{k},\pi}^{\mathbf{T}}+\sum_{k\neq i,j}\left(2B_jC_{ik}+C_{ji}C_{kj}\right)\sigma_{\mathbf{j},\pi}\sigma_{\mathbf{k},\pi}^{\mathbf{T}}+\\
& + \frac{1-\gamma}{\gamma}\left[\sum_{k\neq i,j}\left(2B_iC_{jk}+C_{ij}C_{ki}\right)\overline{\sigma}_{\mathbf{i},\pi}\overline{\sigma}_{\mathbf{k},\pi}^{\mathbf{T}}+\sum_{k\neq i,j}\left(2B_jC_{ik}+C_{ji}C_{kj}\right)\overline{\sigma}_{\mathbf{j},\pi}\overline{\sigma}_{\mathbf{k},\pi}^{\mathbf{T}}\right]+\\
& + \sum_{k<l\neq i,j}\left(C_{ki}C_{lj}+C_{kj}C_{li}\right)\sigma_{\mathbf{k},\pi}\sigma_{\mathbf{l},\pi}^{\mathbf{T}}+\sum_{k\neq i,j}C_{ik}C_{jk}\sigma_{\mathbf{k},\pi}\sigma_{\mathbf{k},\pi}^{\mathbf{T}}+\\
& + \frac{1-\gamma}{\gamma}\left[\sum_{k<l\neq i,j}\left(C_{ki}C_{lj}+C_{kj}C_{li}\right)\overline{\sigma}_{\mathbf{k},\pi}\overline{\sigma}_{\mathbf{l},\pi}^{\mathbf{T}}+\sum_{k\neq i,j}C_{ik}C_{jk}\overline{\sigma}_{\mathbf{k},\pi}\overline{\sigma}_{\mathbf{k},\pi}^{\mathbf{T}}\right]+\\
& + \frac{1-\gamma}{\gamma}\left(\mathbb{A}_{:,i}\right)^{\mathbf{T}}\left[\left(\mathbf{V}\pi_{\infty}\right)\left(\mathbf{V}\pi_{\infty}\right)^{\mathbf{T}}\right]^{-\mathbf{1}}\mathbb{A}_{:,j}+\\
& + \frac{1-\gamma}{\gamma}C_{pi}\overline{\sigma}_{\mathbf{p}}\left(\mathbf{V}\pi_{\infty}\right)^{-\mathbf{1}}\mathbb{A}_{:,j}+\frac{1-\gamma}{\gamma}C_{pj}\overline{\sigma}_{\mathbf{p}}\left(\mathbf{V}\pi_{\infty}\right)^{-\mathbf{1}}\mathbb{A}_{:,i}+\\
& + 2\frac{1-\gamma}{\gamma}B_i\overline{\sigma}_{\mathbf{i},\pi}\left(\mathbf{V}\pi_{\infty}\right)^{-\mathbf{1}}\mathbb{A}_{:,j}+\frac{1-\gamma}{\gamma}\sum_{k\neq i}C_{ik}\overline{\sigma}_{\mathbf{k},\pi}\left(\mathbf{V}\pi_{\infty}\right)^{-\mathbf{1}}\mathbb{A}_{:,j}+\\
& + 2\frac{1-\gamma}{\gamma}B_j\overline{\sigma}_{\mathbf{j},\pi}\left(\mathbf{V}\pi_{\infty}\right)^{-\mathbf{1}}\mathbb{A}_{:,i}+\frac{1-\gamma}{\gamma}\sum_{k\neq j}C_{jk}\overline{\sigma}_{\mathbf{k},\pi}\left(\mathbf{V}\pi_{\infty}\right)^{-\mathbf{1}}\mathbb{A}_{:,i}.
\end{aligned}
\tag{16g}
$$

5 The System of Equations to the Problem with Myopic Portfolio Strategy (No Consumption)

Equation a (independent term):

$$\begin{aligned} A_0' &= -\beta + (1-\gamma)\, r + \left(B_p + \frac{1}{2}A_p^2\right)\sigma_{\mathbf{p}}\sigma_{\mathbf{p}}^{\mathbf{T}} + \sum_{i=1}^{m}\left(C_{pi} + A_p A_i\right)\sigma_{\mathbf{p}}\sigma_{\mathbf{i},\pi}^{\mathbf{T}} + \\ &+ \sum_{i=1}^{m} B_i \sigma_{\mathbf{i},\pi}\sigma_{\mathbf{i},\pi}^{\mathbf{T}} + \sum_{i<j} C_{ij}\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{j},\pi}^{\mathbf{T}} + \frac{1}{2}\sum_{i,j=1}^{m} A_i A_j \sigma_{\mathbf{i},\pi}\sigma_{\mathbf{j},\pi}^{\mathbf{T}} + \\ &+ \frac{1-\gamma}{2\gamma} r^2 \mathbf{1}^{\mathbf{T}} \left[(\mathbf{V}\pi_\infty)(\mathbf{V}\pi_\infty)^{\mathbf{T}}\right]^{-1}\mathbf{1} - \\ &- \frac{1-\gamma}{\gamma} r A_p \overline{\sigma}_{\mathbf{p}} (\mathbf{V}\pi_\infty)^{-1}\mathbf{1} - \frac{1-\gamma}{\gamma} r \sum_{i=1}^{m} A_i \overline{\sigma}_{\mathbf{i},\pi} (\mathbf{V}\pi_\infty)^{-1}\mathbf{1}. \end{aligned} \tag{17a}$$

Equation b (p_t term):

$$\begin{aligned} A_p' &= 2A_p B_p \sigma_{\mathbf{p}}\sigma_{\mathbf{p}}^{\mathbf{T}} + \sum_{i=1}^{m}\left(A_p C_{pi} + 2A_i B_p\right)\sigma_{\mathbf{p}}\sigma_{\mathbf{i},\pi}^{\mathbf{T}} + \\ &+ \frac{1}{2}\sum_{i,j=1}^{m}\left(A_i C_{pj} + A_j C_{pi}\right)\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{j},\pi}^{\mathbf{T}} - \frac{1-\gamma}{\gamma} r \mathbb{B}^{\mathbf{T}}\left[(\mathbf{V}\pi_\infty)(\mathbf{V}\pi_\infty)^{\mathbf{T}}\right]^{-1}\mathbf{1} + \\ &+ \frac{1-\gamma}{\gamma} A_p \overline{\sigma}_{\mathbf{p}} (\mathbf{V}\pi_\infty)^{-1}\mathbb{B} - 2\frac{1-\gamma}{\gamma} r B_p \overline{\sigma}_{\mathbf{p}} (\mathbf{V}\pi_\infty)^{-1}\mathbf{1} + \\ &+ \frac{1-\gamma}{\gamma}\sum_{i=1}^{m} A_i \overline{\sigma}_{\mathbf{i},\pi} (\mathbf{V}\pi_\infty)^{-1}\mathbb{B} - \frac{1-\gamma}{\gamma} r \sum_{i=1}^{m} C_{pi}\overline{\sigma}_{\mathbf{i},\pi}(\mathbf{V}\pi_\infty)^{-1}\mathbf{1}. \end{aligned} \tag{17b}$$

Equation c ($\pi_{i,t}$, $i \in R$, terms):

$$\begin{aligned} A_i' &= A_p \mu_{p,i} + A_p C_{pi}\sigma_{\mathbf{p}}\sigma_{\mathbf{p}}^{\mathbf{T}} + \left(2A_p B_i + A_i C_{pi}\right)\sigma_{\mathbf{p}}\sigma_{\mathbf{i},\pi}^{\mathbf{T}} + \sum_{j\neq i}\left(A_p C_{ji} + A_j C_{pi}\right)\sigma_{\mathbf{p}}\sigma_{\mathbf{j},\pi}^{\mathbf{T}} + \\ &+ \sum_{j=1}^{m}\lambda_{ij}A_j + 2A_i B_i \sigma_{\mathbf{i},\pi}\sigma_{\mathbf{i},\pi}^{\mathbf{T}} + \sum_{j\neq i}\left(A_i C_{ji} + 2A_j B_i\right)\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{j},\pi}^{\mathbf{T}} + \\ &+ \frac{1}{2}\sum_{j,k\neq i}\left(A_j C_{ki} + A_k C_{ji}\right)\sigma_{\mathbf{j},\pi}\sigma_{\mathbf{k},\pi}^{\mathbf{T}} - \frac{1-\gamma}{\gamma} r \mathbf{1}^{\mathbf{T}}\left[(\mathbf{V}\pi_\infty)(\mathbf{V}\pi_\infty)^{\mathbf{T}}\right]^{-1}\mathbb{A}_{:,i} + \\ &+ \frac{1-\gamma}{\gamma}\left[A_p\overline{\sigma}_{\mathbf{p}}(\mathbf{V}\pi_\infty)^{-1}\mathbb{A}_{:,i} - rC_{pi}\overline{\sigma}_{\mathbf{p}}(\mathbf{V}\pi_\infty)^{-1}\mathbf{1} + \sum_{j=1}^{m}A_j\overline{\sigma}_{\mathbf{j},\pi}(\mathbf{V}\pi_\infty)^{-1}\mathbb{A}_{:,i}\right] - \\ &- 2\frac{1-\gamma}{\gamma} r B_i \overline{\sigma}_{\mathbf{i},\pi}(\mathbf{V}\pi_\infty)^{-1}\mathbf{1} - \frac{1-\gamma}{\gamma} r \sum_{j\neq i} C_{ji}\overline{\sigma}_{\mathbf{j},\pi}(\mathbf{V}\pi_\infty)^{-1}\mathbf{1}. \end{aligned} \tag{17c}$$

Equation d (p_t^2 term):

$$
\begin{aligned}
B_p' \;=\;& 2B_p^2 \sigma_{\mathbf{p}} \sigma_{\mathbf{p}}^{\mathbf{T}} + 2\sum_{i=1}^{m} B_p C_{pi} \sigma_{\mathbf{p}} \sigma_{\mathbf{i},\pi}^{\mathbf{T}} + \frac{1}{2}\sum_{i,j=1}^{m} C_{pi} C_{pj} \sigma_{\mathbf{i},\pi} \sigma_{\mathbf{j},\pi}^{\mathbf{T}} + \\
&+ \frac{1-\gamma}{2\gamma} \mathbb{B}^{\mathbf{T}} \left[\left(\mathbf{V}\pi_\infty\right) \left(\mathbf{V}\pi_\infty\right)^{\mathbf{T}} \right]^{-\mathbf{1}} \mathbb{B} + \\
&+ 2\frac{1-\gamma}{\gamma} B_p \overline{\sigma}_{\mathbf{p}} \left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}} \mathbb{B} + \frac{1-\gamma}{\gamma} \sum_{i=1}^{m} C_{pi} \overline{\sigma}_{\mathbf{i},\pi} \left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}} \mathbb{B}.
\end{aligned}
\tag{17d}
$$

Equation e ($\pi_{i,t}^2$, $i \in R$, terms):

$$
\begin{aligned}
B_i' \;=\;& C_{pi} \mu_{p,i} + \frac{1}{2} C_{pi}^2 \sigma_{\mathbf{p}} \sigma_{\mathbf{p}}^{\mathbf{T}} + 2C_{pi} B_i \sigma_{\mathbf{p}} \sigma_{\mathbf{i},\pi}^{\mathbf{T}} + \sum_{j \neq i} C_{pi} C_{ji} \sigma_{\mathbf{p}} \sigma_{\mathbf{j},\pi}^{\mathbf{T}} + 2\lambda_{ii} B_i + \\
&+ \sum_{j \neq i} \lambda_{ij} C_{ji} + 2B_i^2 \sigma_{\mathbf{i},\pi} \sigma_{\mathbf{i},\pi}^{\mathbf{T}} + 2B_i \sum_{j \neq i} C_{ji} \sigma_{\mathbf{i},\pi} \sigma_{\mathbf{j},\pi}^{\mathbf{T}} + \\
&+ \frac{1}{2} \sum_{j,k \neq i} C_{ji} C_{ki} \sigma_{\mathbf{j},\pi} \sigma_{\mathbf{k},\pi}^{\mathbf{T}} + \frac{1-\gamma}{2\gamma} \mathbb{A}_{:,i}^{\mathbf{T}} \left[\left(\mathbf{V}\pi_\infty\right) \left(\mathbf{V}\pi_\infty\right)^{\mathbf{T}} \right]^{-\mathbf{1}} \mathbb{A}_{:,i} + \\
&+ \frac{1-\gamma}{\gamma} C_{pi} \overline{\sigma}_{\mathbf{p}} \left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}} \mathbb{A}_{:,i} + 2\frac{1-\gamma}{\gamma} B_i \overline{\sigma}_{\mathbf{i},\pi} \left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}} \mathbb{A}_{:,i} + \\
&+ \frac{1-\gamma}{\gamma} \sum_{j \neq i} C_{ji} \overline{\sigma}_{\mathbf{j},\pi} \left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}} \mathbb{A}_{:,i}.
\end{aligned}
\tag{17e}
$$

Equation f ($p_t \pi_{i,t}$, $i \in R$, terms):

$$
\begin{aligned}
C_{pi}' \;=\;& 2B_p \mu_{p,i} + 2B_p C_{pi} \sigma_{\mathbf{p}} \sigma_{\mathbf{p}}^{\mathbf{T}} + 4B_p B_i \sigma_{\mathbf{p}} \sigma_{\mathbf{i},\pi}^{\mathbf{T}} + C_{pi}^2 \sigma_{\mathbf{p}} \sigma_{\mathbf{i},\pi}^{\mathbf{T}} + \\
&+ \sum_{j \neq i} \left(2B_p C_{ji} + C_{pi} C_{pj}\right) \sigma_{\mathbf{p}} \sigma_{\mathbf{j},\pi}^{\mathbf{T}} + \sum_{j=1}^{m} \lambda_{ij} C_{pj} + \\
&+ 2B_i C_{pi} \sigma_{\mathbf{i},\pi} \sigma_{\mathbf{i},\pi}^{\mathbf{T}} + \sum_{j \neq i} \left(2B_i C_{pj} + C_{pi} C_{ji}\right) \sigma_{\mathbf{i},\pi} \sigma_{\mathbf{j},\pi}^{\mathbf{T}} + \\
&+ \frac{1}{2} \sum_{j,k \neq i} \left(C_{pj} C_{ki} + C_{pk} C_{ji}\right) \sigma_{\mathbf{j},\pi} \sigma_{\mathbf{k},\pi}^{\mathbf{T}} + \frac{1-\gamma}{\gamma} \left(\mathbb{A}_{:,i}\right)^{\mathbf{T}} \left[\left(\mathbf{V}\pi_\infty\right) \left(\mathbf{V}\pi_\infty\right)^{\mathbf{T}} \right]^{-\mathbf{1}} \mathbb{B} + \\
&+ \frac{1-\gamma}{\gamma} \left[2B_p \overline{\sigma}_{\mathbf{p}} \left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}} \mathbb{A}_{:,i} + C_{pi} \overline{\sigma}_{\mathbf{p}} \left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}} \mathbb{B} + \sum_{j=1}^{m} C_{pj} \overline{\sigma}_{\mathbf{j},\pi} \left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}} \mathbb{A}_{:,i} \right] + \\
&+ 2\frac{1-\gamma}{\gamma} B_i \overline{\sigma}_{\mathbf{i},\pi} \left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}} \mathbb{B} + \frac{1-\gamma}{\gamma} \sum_{j \neq i} C_{ji} \overline{\sigma}_{\mathbf{j},\pi} \left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}} \mathbb{B}.
\end{aligned}
\tag{17f}
$$

Equation g ($\pi_{i,t}\pi_{j,t}$, $i < j \in R$, terms):

$$\begin{aligned}
C'_{ij} = {} & C_{pi}\mu_{p,j} + C_{pj}\mu_{p,i} + C_{pi}C_{pj}\sigma_{\mathbf{p}}\sigma_{\mathbf{p}}^{\mathbf{T}} + \left(2B_iC_{pj} + C_{pi}C_{ij}\right)\sigma_{\mathbf{p}}\sigma_{\mathbf{i},\pi}^{\mathbf{T}} + \\
& + \left(2B_jC_{pi} + C_{pj}C_{ij}\right)\sigma_{\mathbf{p}}\sigma_{\mathbf{j},\pi}^{\mathbf{T}} + \sum_{k\neq i,j}\left(C_{pi}C_{kj} + C_{pj}C_{ki}\right)\sigma_{\mathbf{p}}\sigma_{\mathbf{k},\pi}^{\mathbf{T}} + \\
& + 2\lambda_{ij}B_j + 2\lambda_{ji}B_i + \lambda_{ii}C_{ij} + \lambda_{jj}C_{ij} + \sum_{k\neq i,j}\left(\lambda_{ik}C_{kj} + \lambda_{jk}C_{ki}\right) + \\
& + 2B_iC_{ij}\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{i},\pi}^{\mathbf{T}} + 2B_jC_{ij}\sigma_{\mathbf{j},\pi}\sigma_{\mathbf{j},\pi}^{\mathbf{T}} + \left(4B_iB_j + C_{ij}^2\right)\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{j},\pi}^{\mathbf{T}} + \\
& + \sum_{k\neq i,j}\left(2B_iC_{jk} + C_{ij}C_{ki}\right)\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{k},\pi}^{\mathbf{T}} + \sum_{k\neq i,j}\left(2B_jC_{ik} + C_{ji}C_{kj}\right)\sigma_{\mathbf{j},\pi}\sigma_{\mathbf{k},\pi}^{\mathbf{T}} + \\
& + \sum_{k<l\neq i,j}\left(C_{ki}C_{lj} + C_{kj}C_{li}\right)\sigma_{\mathbf{k},\pi}\sigma_{\mathbf{l},\pi}^{\mathbf{T}} + \sum_{k\neq i,j} C_{ik}C_{jk}\sigma_{\mathbf{k},\pi}\sigma_{\mathbf{k},\pi}^{\mathbf{T}} + \\
& + \frac{1-\gamma}{\gamma}\left(\mathbb{A}_{:,i}\right)^{\mathbf{T}}\left[\left(\mathbf{V}\pi_\infty\right)\left(\mathbf{V}\pi_\infty\right)^{\mathbf{T}}\right]^{-\mathbf{1}}\mathbb{A}_{:,j} + \\
& + \frac{1-\gamma}{\gamma}C_{pi}\overline{\sigma}_{\mathbf{p}}\left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}}\mathbb{A}_{:,j} + \frac{1-\gamma}{\gamma}C_{pj}\overline{\sigma}_{\mathbf{p}}\left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}}\mathbb{A}_{:,i} + \\
& + 2\frac{1-\gamma}{\gamma}B_i\overline{\sigma}_{\mathbf{i},\pi}\left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}}\mathbb{A}_{:,j} + \frac{1-\gamma}{\gamma}\sum_{k\neq i}C_{ik}\overline{\sigma}_{\mathbf{k},\pi}\left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}}\mathbb{A}_{:,j} + \\
& + 2\frac{1-\gamma}{\gamma}B_j\overline{\sigma}_{\mathbf{j},\pi}\left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}}\mathbb{A}_{:,i} + \frac{1-\gamma}{\gamma}\sum_{k\neq j}C_{jk}\overline{\sigma}_{\mathbf{k},\pi}\left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}}\mathbb{A}_{:,i}.
\end{aligned} \tag{17g}$$

6 Figures

Figure 1: Filtered probabilities of being in each of the states. For every point in time, these probabilities are estimated using only the information available before time t. We begin our filter in January 1952. The data sample is from July 1962 to July 2018.

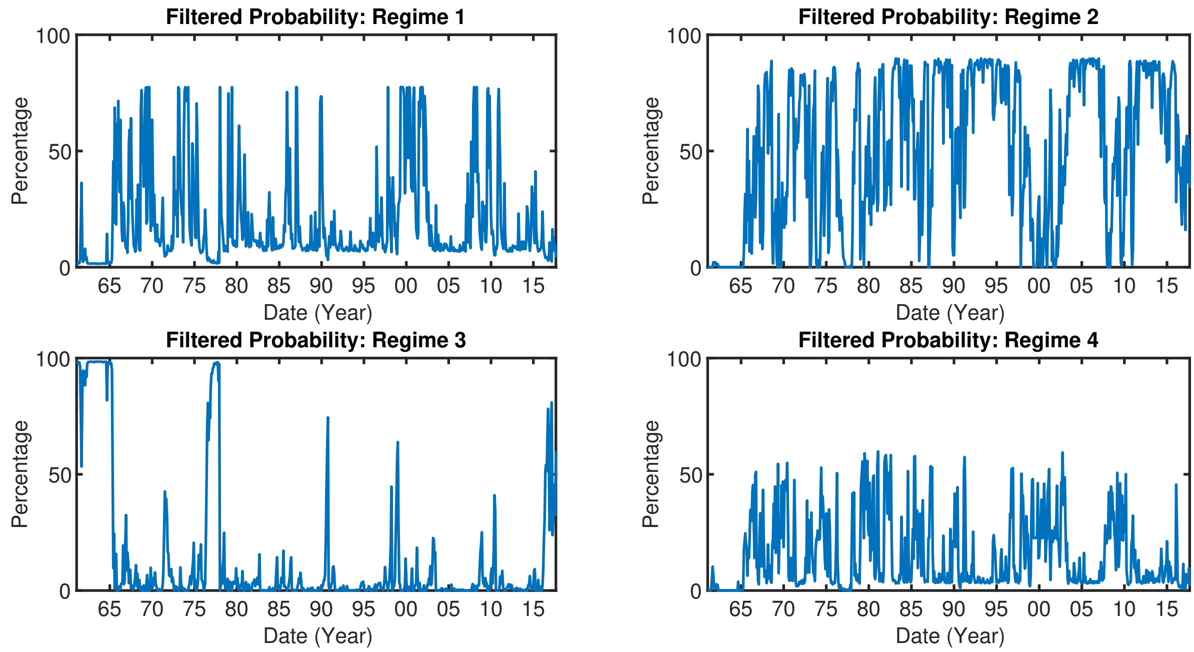


Figure 2: Dynamic strategies for different horizons as given by the model without predictability presented in the paper. Short-selling is allowed. The horizon is measured in years. The current regime is known. Investor's relative risk aversion is set $\gamma = 5$.

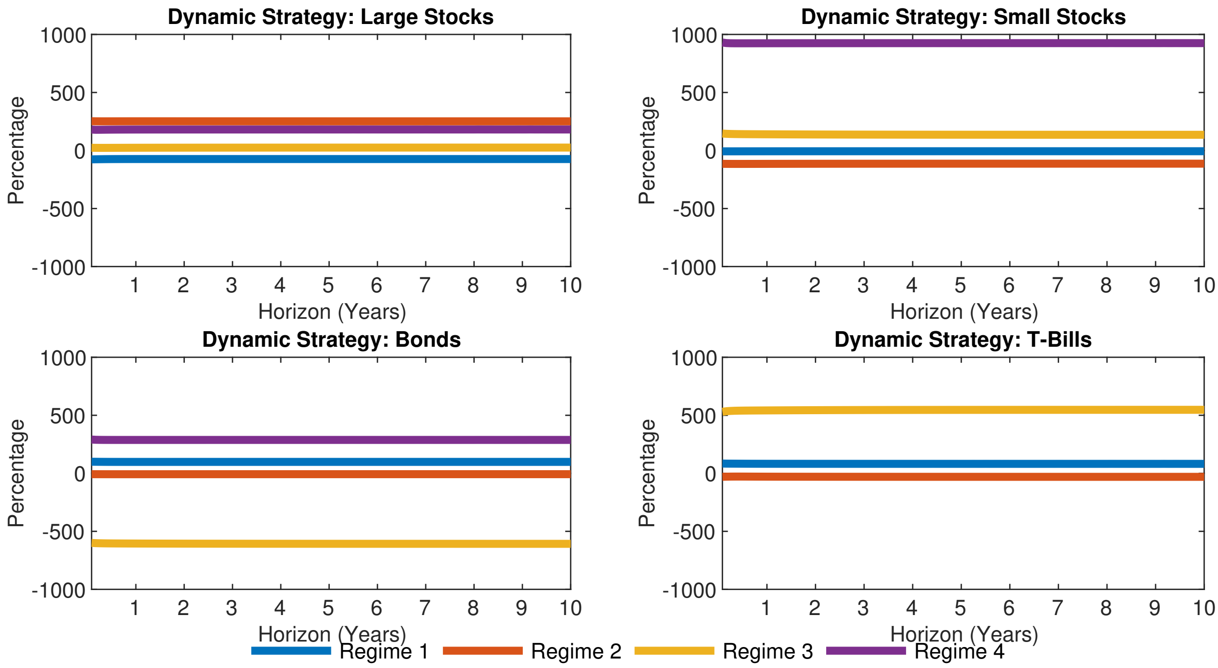


Figure 3: Dynamic strategies for different horizons as given by the model without predictability presented in the paper. Short-selling is allowed. The horizon is measured in years. The current regime is unknown and we plot below the strategies when the current probabilities are either the ergodic ones or equal probabilities for each state. Investor's relative risk aversion is set $\gamma = 5$.

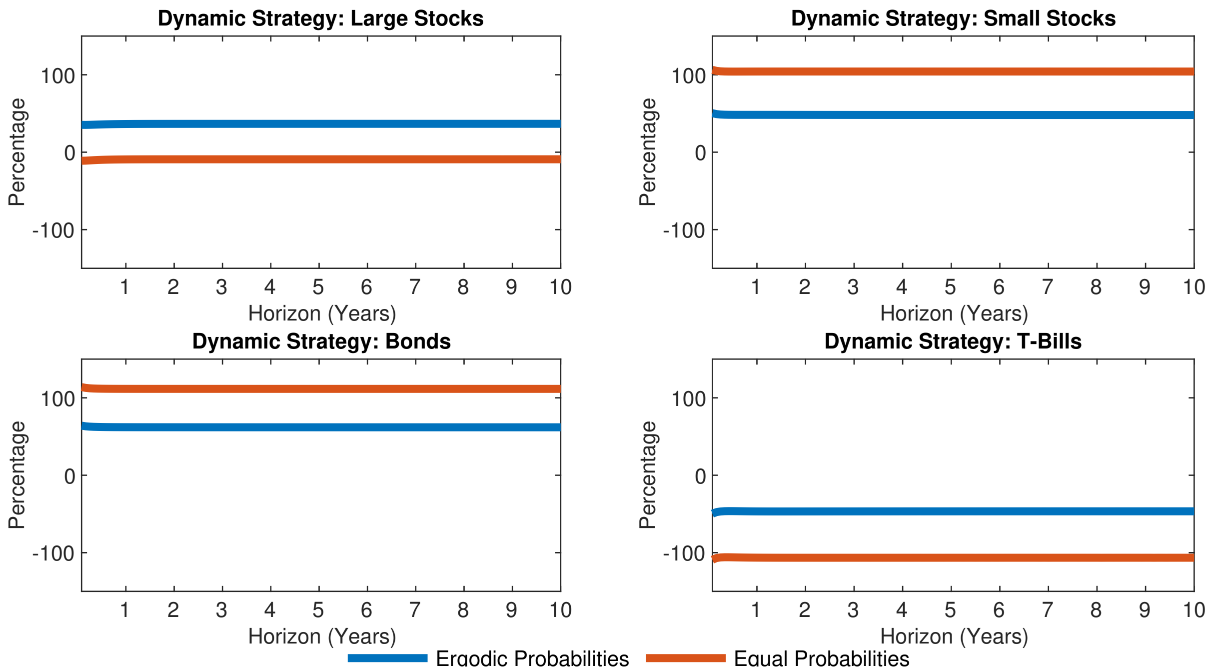
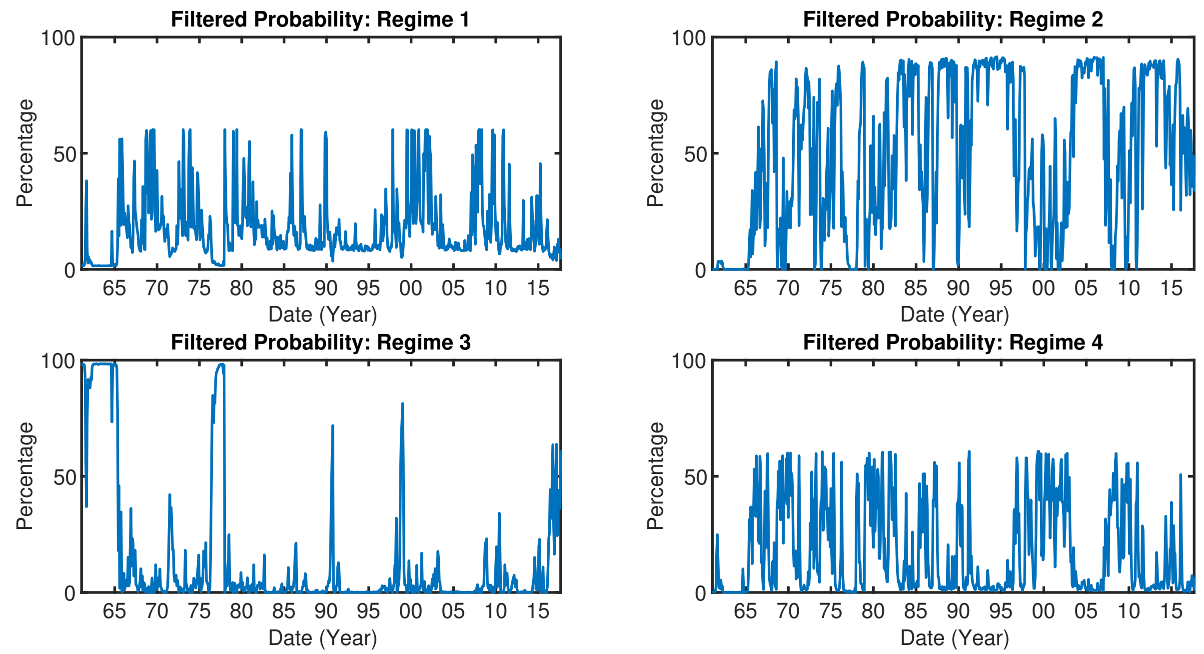


Figure 4: Filtered probabilities of being in each of the states for the problem with predictability. For every point in time, these probabilities are estimated using only the information available before time t. We begin our filter in January 1952. The data sample is from July 1962 to July 2018.



7 Tables

Table 1: Parameter Estimates of the Four-Regime Model with Predictability We report below the parameters for the four state model with predictability. The correlation matrices present the volatilities in their diagonals. We also present the mean expected returns for the assets under each regime considering that the predictor is equal to its mean value in the respective regime. All data is monthly.

Four State Model with Predictability	Large Caps	Small Caps	LT Bonds
Matrix $\mathbb{A}$ Transposed - Regime 1 (Crash)	-2.99%	-4.23%	0.44%
Regime 2 (Slow Growth)	1.18%	0.98%	0.21%
Regime 3 (Bull)	1.21%	1.76%	-0.22%
Regime 4 (Recovery)	4.01%	6.83%	0.56%
Vector $\mathbb{B}$ Transposed	-0.07%	-0.18%	0.28%
Mean Returns - Regime 1 (Crash)	-3.06%	-4.40%	0.80%
Regime 2 (Slow Growth)	1.11%	0.80%	0.56%
Regime 3 (Bull)	1.14%	1.59%	0.13%
Regime 4 (Recovery)	3.93%	6.65%	0.91%
Correlation Matrix - *Regime 1 (Crash)*			
Large Caps	4.92%		
Small Caps	79.15%	7.53%	
LT Bonds	-5.85%	-1.01%	2.61%
Correlation Matrix - *Regime 2 (Slow Growth)*			
Large Caps	2.90%		
Small Caps	72.96%	4.12%	
LT Bonds	16.40%	-2.99%	1.90%
Correlation Matrix - *Regime 3 (Bull)*			
Large Caps	3.22%		
Small Caps	72.41%	3.68%	
LT Bonds	2.51%	-2.31%	0.83%
Correlation Matrix - *Regime 4 (Recovery)*			
Large Caps	3.85%		
Small Caps	20.83%	5.25%	
LT Bonds	24.01%	-12.85%	3.10%

Transition Probabilities	Regime 1	Regime 2	Regime 3	Regime 4
Regime 1 (Crash)	60.21%	0.01%	0.00%	39.78%
Regime 2 (Slow Growth)	7.24%	92.76%	0.00%	0.00%
Regime 3 (Bull)	1.44%	0.00%	98.56%	0.00%
Regime 4 (Recovery)	17.81%	18.38%	3.06%	60.75%
Predictor's Drifts D_p	0.0283	-0.0030	0.0055	-0.0239
Predictor's Volatility Vector σ_p	-0.0177	-0.0072	0.0034	0.0673

Table 2: Volatility Matrix and Ergodic Probabilities We show below the estimated volatility matrix for the probability processes when there are 4 regimes with the three risky assets described in the text and predictability. The estimation procedure is explained in Section 5.1. Data are from January, 1953 to July, 2018. The table also reports the ergodic (unconditional) probabilities and the average duration of each regime.

	Z_1	Z_2	Z_3	Z_p	Ergodic Prob.	Average Duration
Regime 1	-5.31%	-2.15%	0.42%	0.03%	15%	3 months
Regime 2	4.11%	-0.50%	-0.64%	0.02%	38%	14 months
Regime 3	0.91%	1.11%	-0.24%	-0.01%	32%	69 months
Regime 4	0.29%	1.55%	0.46%	0.01%	15%	3 months