

Approximate Analytical Solutions for Consumption/Investment Problems under Recursive Utility and Finite Horizon.*

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Abstract

We study the asset allocation and consumption decisions of an investor with recursive utility and a finite investment horizon. We provide an approximate analytical solution under a stochastic investment opportunity set. The solution becomes exact when the elasticity of intertemporal substitution is equal to one or under a constant opportunity set. We show that this elasticity impacts both consumption and portfolio strategies, indicating the importance of disentangling intertemporal substitution from risk aversion. The investor's horizon also plays a crucial role in optimal policies and the usual infinite horizon framework is inappropriate for investors having short- or medium-term horizons. Moreover, the infinity horizon problem reveals the existence of conditions on the preference parameters for solutions to exist while the absence of a bequest motive in the finite horizon problem will impose another condition on risk parameters.

JEL classification: G11, C02

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1 Introduction

Asset allocation occupies a central position in finance. Since the seminal contributions of Markowitz (1952), Samuelson (1969) and Merton (1971), the standard *time additive* problem in which investors maximize the total discounted utility from future consumption and/or from terminal wealth has been the focus of most academic work. When the investor is endowed with a constant relative risk aversion (CRRA) power utility, exact analytical solutions under finite investment horizon are known for several realistic settings featuring complete or incomplete markets (see the review in Brandt (2009) or Wachter (2010)). Alternatively, simulation-based methods have been applied when the setting permitted, as in Detemple et al.(2003) or in Brandt et al.(2005).

Recursive preferences introduced by Kreps and Porteus (1978) and Epstein and Zin (1989) have been used extensively in asset pricing models to relax time additivity in intertemporal decision making under uncertainty. Recently, Bansal and Yaron (2004) and Bonomo et al. (2011) showed that long-run risk asset pricing models with recursive utility fit financial data much better than models based on time-additive preferences.

Recursive preferences have also been used for portfolio choice, either in the discrete-time setting of Epstein and Zin (1989) or the stochastic differential recursive utility introduced by Duffie and Epstein (1992a).¹ A key advantage to these preferences is the prominent role that can be given to the elasticity of intertemporal substitution (EIS), independently of risk aversion. A lively debate exists in the asset pricing literature as to the right value of the EIS for models with recursive utility to fit the data. As shown by some authors, a unitary value of the EIS is pivotal since the implications for asset pricing change dramatically above and below this value.

In this paper we aim at finding optimal analytical solutions for consumption and portfolio allocation in a very general setting where investors have recursive utility over intermediate consumption and terminal wealth in a finite-horizon and incomplete-markets setting. Only partial solutions exist for this problem. Exact solutions to the optimal portfolio problem have been found with a unitary value for the EIS (Schroder and Skiadas (1999) and Zhu (2006) under complete markets, and Campbell et al. (2004) under incomplete markets and an infinite horizon). Otherwise, approximate solutions have been found for an infinite investment horizon. Campbell and Viceira (1999) use a log-linear approximation of the consumption-to-wealth ratio

¹This class nests as special cases the power utility (with constant relative risk aversion) and the log-utility (with myopic asset allocation decisions).

to obtain an approximate solution under incomplete markets. Hansen et al. (2008) expand the value function around the solution obtained with a unitary value for the EIS. For general values of the EIS, no closed-form solutions are available for dynamic asset allocation problems over finite horizons. Previewing our detailed findings, we confirm that horizon effects may be sizable and that the EIS plays a crucial role in the optimal portfolio allocation.

We start by deriving our approximate analytical solution in a simple environment where the investor allocates his wealth between a risky asset and a safe asset. This is the same investment framework as in Kim and Omberg (1996) and Wachter (2002), where in addition the risk premium varies linearly with a given predictor. We are therefore in an incomplete market setting. The horizon is finite and our preferences, represented by a stochastic differential recursive utility from intermediate consumption and terminal wealth, generalize the HARA utility considered in the latter papers. Our solution generalizes these former contributions in several ways. While Kim and Omberg (1996) provides an analytical solution to the portfolio problem for HARA utility over terminal wealth, we consider recursive utility over both intermediate consumption and terminal wealth. Wachter (2002) provides analytical formulas for intermediate consumption problems and power expected utility but assumes a perfect negative correlation between the risky asset returns and the predictor variable, therefore reducing the setting to complete markets.

To obtain this solution, we approximate the logarithm of the consumption-to-wealth ratio around a particular point of the state-variable space instead of its long-run mean as in previous papers. To assess the accuracy of our approximate solution for arbitrary values of the EIS, we verify that for long investment horizons the solution converges towards the solution proposed for an infinite horizon by Campbell et al. (2004). In a continuous time setting with infinite horizon, Campbell et al. (2004) approximate the consumption-to-wealth ratio by a linear function of the *log* consumption to wealth ratio with *constant* coefficients. An analytical solution is therefore possible. However, a similar approach with a finite horizon will make these coefficients in general time-dependent, making it impossible to solve for the optimal portfolio in closed-form. We also confirm that our approximate solution is very close to the exact analytical solution proposed by Kraft et al. (2013). They choose the particular configuration of preference parameters that cancel out the nonlinear terms in the Bellman equation and therefore lead to an analytical solution, with of course the unfortunate consequence of tying again the parameter of risk aversion to the EIS. We also confirm that the optimal portfolio allocation depends on ψ through the hedging demands but not through the myopic portfolio, as shown by Bhamra and Uppal (2006) in a

three-date discrete-time model for a similar stochastic investment setting.

We take advantage of these approximate analytical solutions to measure empirically the optimal consumption and investment decisions in a much richer model with stocks and bonds as risky assets, a stochastic interest rate and predictable risk premia by the dividend yield. We estimate the parameters of the model and show that the consumption-to-wealth ratio is almost flat relative to the investment horizon whenever the EIS is greater than 1 since the investor reduces her current consumption to take better advantage of future opportunities. On the other hand, the investor tends to consume a larger fraction of her wealth whenever the EIS is less than 1, especially near the end of the horizon. Also, values of the EIS greater than 1 tend to make investors more aggressive in that the total investment in risky assets increases substantially. Again, these findings offer an interesting perspective on the ongoing debate regarding the estimated value of the EIS in the asset pricing literature. A value greater than one is necessary to explain stylized facts about asset returns with a long-run risk model. However, irrespective of the value of the EIS, the solution provided by the infinite horizon case tends to overstate the horizon effects for short- and medium-term investors.

The most related paper to this one is Xing (2017), which studies the same problem with recursive utility. However, this paper finds analytical solutions and discusses some important issues not present in Xing (2017). First, we focus on how horizon and the EIS impact optimal strategies. Second, we find relevant conditions under constant opportunities and when there is no motive for bequest. Under a constant opportunity setting, our derivations show that the existence of solutions when letting the horizon go to infinity imposes restrictions on the preference parameters. These restrictions put limits on the impatience of the investor and on its propensity to postpone consumption. Of course these restrictions are also dependent on the investment environment, namely the interest rate and the Sharpe ratio of the risky asset(s). These restrictions are interesting in light of the calibrated values used for preference parameters in recent long-run risk studies and extend the previous literature. Third, in the case of no utility for bequest, we highlight that ψ and γ should be on opposite sides of one, otherwise we are not able to find an adequate value function to the problem.

The rest of the paper is organized as follows. In Section 2, we present a model with a constant interest rate. We first consider a simplified setting to derive an explicit solution when the EIS is equal to 1 and then introduce our approximate solution when it is different from 1. In section 3 we develop a more general setting with stocks and bonds and a stochastic interest rate, and

report the calibration of the parameters as well as the optimal portfolio shares and consumption ratios for various configurations of the preference parameters and horizons. Section 4 concludes. In the appendix, we collect several mathematical developments. We have also created an internet appendix with further demonstrations, graphs and discussions.

2 Optimal Portfolio Choice

2.1 The Economy

Consider a frictionless and arbitrage-free financial market where trading takes place continuously. A single risky asset and a riskless asset are available for trade. We denote the short-term riskless asset price at time t as M_t and assume that it evolves according to the following process:

$$\frac{dM_t}{M_t} = r dt. \quad (1)$$

The assumption of a constant instantaneous interest rate will be relaxed in our more general application in Section 3. The risky asset price is denoted by S_t and it is assumed to have the following dynamics:

$$\frac{dS_t}{S_t} = (r + q_{S,t}) dt + \sigma_S dZ_{S,t}, \quad (2)$$

where $q_{S,t}$ denotes the time-varying expected excess return (risk premium) on the risky asset, and σ_S is its constant volatility. Following Kim and Omberg (1996) and Wachter (2002), amongst many others, we assume that the risk premium varies linearly with a given predictor P_t in the following manner:

$$q_{S,t} = b_{S,0} + b_{S,1}P_t, \quad (3)$$

where $b_{S,j}$ is constant for $j = 0, 1$.

The predictor is assumed to follow an Ornstein-Uhlenbeck process:²

$$dP_t = \kappa_p (\bar{P} - P_t) dt + \sigma_p dZ_{P,t}, \quad (4)$$

where the parameter κ_p is the speed of mean-reversion, \bar{P} is the long-run mean and the volatility

²This is the continuous-time version of an autoregressive process of order one in discrete-time.

of the innovation in the predictor's dynamics is given by σ_p . The correlation between the risky asset price process and the predictor is denoted as ρ_{SP} and is left unconstrained. As a consequence, our market is incomplete since we have two sources of risk and only one traded risky asset.

The investor chooses her optimal consumption plan C optimally at time t . The variable C_t is defined as the rate of consumption at instant t , such that the investor consumes $C_t dt$ over the time interval from t to $t + dt$. She invests a fraction $\alpha_{S,t}$ of her total wealth W_t in the risky asset and the rest in the riskless asset. The wealth dynamics will thus evolve as:

$$\begin{aligned} dW_t &= (1 - \alpha_{S,t}) W_t \frac{dM_t}{M_t} + \alpha_{S,t} W_t \frac{dS_t}{S_t} - C_t dt \\ &= W_t r dt + W_t (\alpha_{S,t} q_{S,t} dt + \alpha_{S,t} \sigma_S dZ_{S,t}) - C_t dt. \end{aligned} \quad (5)$$

The investor obtains her felicity from intermediate consumption and possibly from terminal wealth (bequest), with preferences represented by a continuous-time recursive utility function in the spirit of Duffie and Epstein (1992a):

$$J_t = E_t \left[\int_{u=t}^T f(C_u, J_u) du + \epsilon \frac{W_T^{1-\gamma}}{1-\gamma} \right], \quad (6)$$

in which ϵ controls for the presence ($\epsilon = 1$) or the absence ($\epsilon = 0$) of bequest. Time T is the investment horizon of the investor. While we will be mostly interested in the finite horizon problem, we will still analyze the infinite horizon problem for comparison with the existing literature by letting T tend to infinity.

In equation (6), $f(C, J)$ is a normalized aggregator of the current rate of consumption and continuation utility that takes the following form:

$$f(C, J) = \frac{\beta}{1 - \frac{1}{\psi}} (1 - \gamma) J \left\{ \left[\frac{C}{[(1 - \gamma) J]^{\frac{1}{1-\gamma}}} \right]^{1 - \frac{1}{\psi}} - 1 \right\}. \quad (7a)$$

In equation (7a), $\beta > 0$ is the rate of time preference. The parameter γ controls for investor's attitudes over the states of the economy, while ψ is related to investor's consumption choices over time. We will usually refer to them as the risk aversion and the elasticity of intertemporal substitution (EIS) parameters, respectively. This continuous-time recursive utility function gives rise to a homogeneous value function of degree $(1 - \gamma)$ in wealth (while in the discrete-

time version of our model the value function would be linear in wealth).

There are two special cases of the normalized aggregator given by equation (7a): $\psi = \frac{1}{\gamma}$ and $\psi = 1$. The first case corresponds to the standard, time-additive power utility function.³ In the second case, interpreted as the limit when $\psi \rightarrow 1$, the aggregator takes the following limit form:

$$\psi = 1 \quad \rightarrow \quad f(C, J) = \beta (1 - \gamma) J \ln \left\{ \frac{C}{[(1 - \gamma) J]^{\frac{1}{1-\gamma}}} \right\}. \quad (7b)$$

This case is important because it allows for an exact analytical solution even under our stochastic setting. Furthermore, it represents the singular situation in which investors choose consumption myopically, which means that the optimal consumption-to-wealth ratio will not depend on the investment opportunity set.⁴ The general case, in which $\psi \neq 1$ and γ is an independent parameter, does not have an exact solution in closed form, unless we consider the opportunity set to be constant.⁵ The goal is then to obtain exact solutions when available and an approximate analytical solution in the most general case.

2.2 Optimal Consumption and Portfolio Decisions

In this subsection, we derive the general Bellman equation for the stochastic setting just described. Since it does not admit in general an analytical solution, we look first at a particular case where such an exact solution exists: the case in which $\psi = 1$. Then we will present an approximate analytical solution for the general case with bequest in subsection 2.4. In subsection 2.5, we analyze interesting features of the constant opportunity setting, which also admits an exact solution in closed form. In the text, we focus on the case with bequest, leaving for the appendices some relevant issues that arise when the investor sees no reasons for leaving a bequest.

Duffie and Epstein (1992a,b) have shown that the standard Bellman principle of optimality applies to this continuous-time version of the recursive utility function. For our stochastic setting, the Bellman equation is given by:

³Note that the resulting formulation may not take the standard power utility form, but will imply the same underlying preferences, and hence the same consumption and asset allocation choices. See further details in our online appendix to this paper.

⁴When $\psi > 1$, consumption falls relative to wealth when investment opportunities improve (the substitution effect dominates) while when $\psi < 1$, consumption will rise (the income effect dominates). The singular case in which $\psi = 1$ makes both effects cancel out; see for example Wachter (2010).

⁵We discuss the constant opportunity set in subsection 2.5 and in appendices A and B.

$$0 = \sup_{\{C_t, \alpha_{S,t}\}} \left[\begin{array}{l} f(C_t, J_t) + \frac{\partial J_t}{\partial t} + J_w \{W_t r + W_t [\alpha_{S,t} (b_{S,0} + b_{S,1} P_t)] - C_t\} + \\ + J_p \kappa_p (\bar{P} - P_t) + \frac{1}{2} W_t^2 J_{ww} \alpha_{S,t}^2 \sigma_S^2 + \frac{1}{2} J_{pp} \sigma_p^2 + J_{wp} W_t \rho_{SP} \alpha_{S,t} \sigma_S \sigma_p \end{array} \right], \quad (8)$$

where $f(C, J)$ is given by equation (7a) if $\psi \neq 1$ or (7b) if $\psi = 1$. The subscripts denote partial derivatives, except t , which denotes the variable value at time t . The function $V(W_t, P_t, t) \equiv \sup_{\{C_t, \alpha_{S,t}\}} [J(W_t, P_t, t)]$ is the value function for this problem.

The first-order condition for consumption is given by:

$$V_w = \frac{\partial f(C_t, V_t)}{\partial C_t} = \beta (1 - \gamma)^{\frac{1}{1-\gamma}} V_t^{\frac{1-\gamma}{1-\gamma}} C_t^{-\frac{1}{\psi}}. \quad (9a)$$

Solving for the optimal consumption strategy, one finds that:

$$C_t = \beta^\psi V_w^{-\psi} [(1 - \gamma) V_t]^{\frac{1-\psi\gamma}{1-\gamma}}, \quad (9b)$$

which reduces to $C_t = \beta V_w^{-1} V_t (1 - \gamma)$ when $\psi = 1$. We can insert equation (9b) into (7a) and (7b) to obtain:

$$f(V_t) = \frac{1}{1 - \frac{1}{\psi}} \left\{ \beta^\psi V_w^{1-\psi} [(1 - \gamma) V_t]^{\frac{1-\psi\gamma}{1-\gamma}} - \beta (1 - \gamma) V_t \right\} \quad \psi \neq 1, \quad (10a)$$

$$f(V_t) = \beta (1 - \gamma) V_t \ln \left\{ \beta V_w^{-1} [(1 - \gamma) V_t]^{\frac{-\gamma}{1-\gamma}} \right\} \quad \psi = 1. \quad (10b)$$

The first-order condition for the risky asset position is given by:

$$V_w (b_{S,0} + b_{S,1} P_t) + V_{ww} W_t \alpha_{S,t} \sigma_S^2 + V_{wp} \rho_{SP} \sigma_S \sigma_p = 0, \quad (11)$$

which can be rewritten as:

$$\alpha_{S,t} = -\frac{V_w}{V_{ww} W_t} \frac{b_{S,0} + b_{S,1} P_t}{\sigma_S^2} - \frac{V_{wp}}{V_{ww} W_t} \frac{\rho_{SP} \sigma_p}{\sigma_S}. \quad (12)$$

We find the usual decomposition of the portfolio share into a mean-variance myopic demand and an intertemporal hedging demand due to stochastic variations in the opportunity set. This last demand disappears in a constant investment opportunity set or, as we will confirm, when $\gamma = 1$. We substitute the optimal expressions for consumption and portfolio weights into

equation (8) to obtain the final expression for the Bellman equation:

$$0 = f(V_t) + \frac{\partial V_t}{\partial t} + V_p \kappa_p (\bar{P} - P_t) + \frac{1}{2} V_{pp} \sigma_p^2 + V_w W_t r - \beta^\psi V_w^{1-\psi} [(1-\gamma) V_t]^{\frac{1-\psi\gamma}{1-\gamma}} - \frac{1}{2} \frac{V_w^2}{V_{ww}} \left(\frac{b_{S,0} + b_{S,1} P_t}{\sigma_S} \right)^2 - \frac{V_w V_{wp}}{V_{ww}} \frac{b_{S,0} + b_{S,1} P_t}{\sigma_S} \rho_{SP} \sigma_p - \frac{1}{2} \frac{V_{wp}^2}{V_{ww}} \rho_{SP}^2 \sigma_p^2. \quad (13)$$

This equation does not admit, in general, an analytical solution due to the presence of nonlinear terms. For the particular case of unitary elasticity of intertemporal substitution ($\psi = 1$), an exact solution exists. We derive it in the following subsection and use the findings to motivate the original approximate solution we suggest for the general case.

2.3 The Exact Analytical Solution when $\psi = 1$

Campbell et al.(2004) find an exact explicit solution under incomplete markets which applies only for the infinite horizon problem. Following Schroder and Skiadas (1999) and Zhu (2006), we provide hereafter the explicit solution under finite horizon and incomplete markets (details of the solution are presented in appendix C).

The exact form of the value function that solves the problem when $\psi = 1$ is equal to:

$$V(W_t, P_t, \tau) = \frac{W_t^{1-\gamma}}{1-\gamma} \exp \left\{ (1-\gamma) \left[A_1(\tau) + A_2(\tau) P_t + \frac{A_3(\tau)}{2} P_t^2 \right] \right\}, \quad (14)$$

in which $A_i(\tau)$, $i = 1, 2, 3$, are deterministic functions of the time remaining until the investor's investment horizon ($\tau \equiv T - t$) and depend on the model parameters for the investment opportunities and preferences. They solve a system of ordinary differential equations with boundary conditions $A_i(0) = 0$ for all i .

The optimal consumption and portfolio policies are obtained by substituting solution (14) into equations (9b) and (12):

$$\frac{C_t}{W_t} = \beta, \quad (15a)$$

$$\alpha_{S,t} = \underbrace{\frac{1}{\gamma} \frac{b_{S,0} + b_{S,1} P_t}{\sigma_S^2}}_{\text{Myopic Portfolio}} + \underbrace{\frac{1-\gamma}{\gamma} \frac{[A_2(\tau) + A_3(\tau) P_t] \rho_{SP} \sigma_p}{\sigma_S}}_{\text{Hedging Demand}}. \quad (15b)$$

The consumption policy is constant, *i.e.* the investor consumes a constant fraction of her

wealth each period which is equal to the time preference parameter. The portfolio strategy is linear in the predictor as often put forward in similar affine settings (see Kim and Omberg (1996)). The infinite horizon solution can be seen as a special case of the above solution, applying the limit when $\tau \rightarrow \infty$. This solution is then exactly the one obtained by Campbell et al. (2004).

The solution for the so-called log-investor is nested in the general solution above. Her portfolio policy will include just the myopic demand because the hedging demand vanishes.⁶

2.4 An Approximate Analytical Solution when $\psi \neq 1$

For the general case with a non-unitary elasticity of intertemporal substitution, the Bellman equation is nonlinear and cannot be solved in closed form. Campbell and Viceira (1999) provided an approximation for the discrete time case in which the log consumption ratio is approximated around its unconditional mean. Campbell et al. (2004) show how such an approximation can work in a continuous-time setting. A crucial ingredient for this approximation to work is that the investor's investment horizon is infinite. Under such an assumption, the consumption-to-wealth ratio is written as a linear function of the log consumption-to-wealth ratio and the coefficients of this linear function are *constant*.⁷ When the horizon is finite, they will in general depend upon the time to the investment horizon and this will make the solution offered by the approximation less tractable. We are able to provide an approximate analytical solution which works under finite investment horizon. At the onset, note that some authors reached an exact analytical solution by choosing the preferences parameters such that the nonlinear terms in the Bellman equation cancel out (see Kraft et al. (2013)). This has the unfortunate consequence of tying again the parameter of risk aversion to the EIS, which cancels the disentangling offered by recursive utility.

Based on the solution obtained under the case of unitary intertemporal elasticity of substitution, we guess that the value function has the following form:

$$V(W_t, P_t, \tau) = I(P_t, \tau) \frac{W_t^{1-\gamma}}{1-\gamma}, \quad (16)$$

⁶Proving this last result is not as simple as just substituting $\gamma = 1$ in (15b) Instead, we need to analyze how the products $(1-\gamma)A_i(\tau)$ go in the limit when $\gamma \rightarrow 1$ because the $A_i(\tau)$ terms also depend on γ . The easiest way to do that is doing a change of variables in the original system (call $B_i(\tau) = (1-\gamma)A_i(\tau)$), which will then prove that $(1-\gamma)A_i(\tau) \rightarrow 0$ for $i = 1, 2, 3$. Alternatively, one can note that when gamma goes to 1, the system does not explode, meaning that it will still provide finite solutions for all coefficients.

⁷See equation (B.5) page 2212 of Campbell et al. (2004).

with terminal condition $I(P_T, 0) = 1$ (due to the presence of bequest). We substitute this candidate solution into the Bellman equation given by equation (13) to obtain:

$$0 = \frac{\beta^\psi}{\psi-1} I^{1-\gamma} - \frac{\beta^\psi}{\psi-1} - \frac{1}{1-\gamma} \frac{\partial I}{\partial \tau} I^{-1} + \frac{1}{1-\gamma} \frac{\partial I}{\partial \bar{P}} I^{-1} \kappa_p (\bar{P} - P_t) + \frac{1}{2} \frac{1}{1-\gamma} \frac{\partial^2 I}{\partial P^2} I^{-1} \sigma_p^2 + r + \frac{1}{2\gamma} \left(\frac{b_{S,0} + b_{S,1} P_t}{\sigma_S} \right)^2 + \frac{1}{\gamma} \frac{\partial I}{\partial \bar{P}} I^{-1} \frac{b_{S,0} + b_{S,1} P_t}{\sigma_S} \rho_{SP} \sigma_p + \frac{1}{2\gamma} \left(\frac{\partial I}{\partial \bar{P}} \right)^2 I^{-2} \rho_{SP}^2 \sigma_p^2. \quad (17)$$

Therefore, the consumption-to-wealth ratio is given by:

$$\frac{C_t}{W_t} = \beta^\psi I^{\frac{1-\psi}{1-\gamma}}. \quad (18)$$

Looking at the HJB equation, one remarks that the non-linearities are, as usual, brought about by the consumption-to-wealth ratio. Therefore, it is the essential variable to approximate or, in other words, to linearize. Therefore, we suggest to approximate the consumption-to-wealth ratio around the long-run mean of the predictor. Due to the well-known persistence of usual predictors and thus the low volatility of their innovations, such an approximation is likely to be accurate even when it stops at the first order. More precisely, the approximation we suggest is:

$$\exp\left(\ln \frac{C_t}{W_t}\right) \approx \exp\left(\ln \frac{C_t}{W_t}\right)_{P_t=\bar{P}} + \exp\left(\ln \frac{C_t}{W_t}\right)_{P_t=\bar{P}} \left[\ln \frac{C_t}{W_t} - \left(\ln \frac{C_t}{W_t}\right)_{P_t=\bar{P}} \right] \quad (19)$$

Using such an approximation, we can make the usual guess for the value function and recover thus a tractable (explicit indeed) solution. We thus guess:

$$I(P_t, \tau) = \exp\left\{ (1-\gamma) \left[A_1(\tau) + A_2(\tau) P_t + \frac{A_3(\tau)}{2} P_t^2 \right] \right\}, \quad (20)$$

and the value function can therefore be written as:

$$V(W_t, P_t, \tau) = \frac{W_t^{1-\gamma}}{1-\gamma} \exp\left\{ (1-\gamma) \left[A_1(\tau) + A_2(\tau) P_t + \frac{A_3(\tau)}{2} P_t^2 \right] \right\},$$

with terminal condition $V(W_t, P_t, 0) = \frac{W_t^{1-\gamma}}{1-\gamma}$.

By substituting this expression into (17) and collecting terms, we solve for $A_i(\tau)$, $i = 1, 2, 3$, in a system of ordinary differential equations with boundary conditions equal to zero since the value function should converge to the power utility of wealth irrespectively of state-variable values at the final horizon. Appendix D details the system of ordinary differential equations.

The optimal choices are given by:

$$\frac{C_t}{W_t} = \beta^\psi \exp \left\{ (1 - \psi) \left[A_1(\tau) + A_2(\tau) \bar{P} + \frac{A_3(\tau)}{2} \bar{P}^2 \right] \right\} \left\{ 1 + (1 - \psi) \left[A_2(\tau) (P_t - \bar{P}) + \frac{A_3(\tau)}{2} (P_t^2 - \bar{P}^2) \right] \right\} \quad (21)$$

$$\alpha_{S,t} = \frac{1}{\gamma} \frac{b_{S,0} + b_{S,1} P_t}{\sigma_S^2} + \frac{1 - \gamma}{\gamma} \frac{[A_2(\tau) + A_3(\tau) P_t] \rho_{SP} \sigma_P}{\sigma_S}. \quad (22)$$

Notice that the consumption-to-wealth ratio depends stochastically upon the investment horizon through the time-varying coefficients $A_i(\tau)$. From the system of equations for the coefficients in the appendix, we see that they will depend on ψ . As a consequence, the optimal portfolio allocation will also depend on ψ through the hedging demands (but not through the myopic portfolio). This finding confirms the work of Bhamra and Uppal (2006). In a three-date discrete-time model exercise, with a stochastic investment opportunity set and Epstein and Zin (1989) recursive utility, they derive an optimal portfolio that depends on the elasticity of intertemporal substitution parameter through the hedging demands only. Finally, comparing the systems in the respective appendices, we see that the solution is continuous when $\psi = 1$ not only for the value function but also for the optimal choices.

In the infinite horizon case, when $\tau \rightarrow \infty$, the value function, consumption-to-wealth ratio and the portfolio strategy take the same form as above, but with constant coefficients. It is important to highlight that this infinite horizon approximate solution does not nest the log-linear solution of Campbell et al. (2004) since the approximation is made around a different point. The value function and optimal policies will have the same functional form as ours, but with different coefficients.⁸ We do compare both solutions in Section 3, under a more general stochastic setting.

2.5 Constant Opportunity Set

We now focus on a particular case nested in the stochastic framework analyzed above (and notice that the exact solutions found in this subsection are nested in the previous solutions). In this context, the single stochastic state variable is wealth, with a constant interest rate and a constant risk premium q_S (which means no predictability). Under these assumptions, the

⁸We will use this solution as a benchmark to compare with ours, but we can anticipate that both will provide virtually the same answers under realistic values of ψ and state variable(s).

Bellman equation (13) simplifies in the following way:

$$0 = f(V_t) + \frac{\partial V_t}{\partial t} + V_w W_t r - \beta^\psi V_w^{1-\psi} [(1-\gamma) V_t]^{\frac{1-\psi\gamma}{1-\gamma}} - \frac{1}{2} \left(\frac{q_S}{\sigma_S} \right)^2 \frac{V_w^2}{V_{ww}}. \quad (23)$$

As the aggregator $f(V_t)$ takes two different forms, we need to analyze the cases $\psi = 1$ and $\psi \neq 1$ separately.

2.5.1 The Exact Analytical Solution when $\psi = 1$

The value function that exactly solves equation (23) when $\psi = 1$ and with bequest is:⁹

$$V(W_t, \tau) = e^{k(1-\gamma)(1-e^{-\beta\tau})} \frac{W_t^{1-\gamma}}{1-\gamma}, \quad (24)$$

in which $\tau = T - t$ is the remaining time till the final horizon and $k = \ln\beta + \frac{r}{\beta} + \frac{q_S^2}{2\beta\gamma\sigma_S^2} - 1$. Notice that $V(W_T, 0) = \frac{W_T^{1-\gamma}}{1-\gamma}$.

The optimal policies are given by:

$$\frac{C_t}{W_t} = \beta, \quad (25a)$$

$$\alpha_{S,t} = \frac{1}{\gamma} \frac{q_S}{\sigma_S^2}. \quad (25b)$$

Observe that the horizon does not impact at all the optimal choices, which are constant during the investment horizon. The consumption policy is equal to the value found in the previous literature, as for example in Campbell et al. (2004) under an infinite horizon. When $\psi = 1$, the investor consumes myopically, in the sense that the investment opportunity set will not affect her consumption strategy. The optimal portfolio rule is also a well-known result. The myopic strategy is proportional to the risk premium and inversely proportional to the relative risk aversion coefficient.

2.5.2 The Exact Analytical Solution when $\psi \neq 1$

In this case, the value function that exactly solves equation (23) is given by:

$$V(W_t, \tau) = h(\tau) \frac{W_t^{1-\gamma}}{1-\gamma}, \quad (26a)$$

⁹We show in appendices A and B all the detailed proofs of the results presented. In appendix B.1, we put forward important issues in the infinite horizon problem.

where:

$$h(\tau) = \left\{ \left[1 + \frac{\beta^\psi}{\left(r + \frac{q_S^2}{2\gamma\sigma_S^2}\right)(\psi - 1) - \psi\beta} \right] e^{\left[\left(r + \frac{q_S^2}{2\gamma\sigma_S^2}\right)(\psi - 1) - \psi\beta\right]\tau} - \frac{\beta^\psi}{\left(r + \frac{q_S^2}{2\gamma\sigma_S^2}\right)(\psi - 1) - \psi\beta} \right\}^{\frac{1-\gamma}{\psi-\gamma}}. \quad (26b)$$

The optimal consumption policy is:

$$\frac{C_t}{W_t} = \beta^\psi h^{\frac{1-\psi}{1-\gamma}}. \quad (27)$$

Notice that the consumption strategy is now time-varying and depends, through the function h , on market conditions.¹⁰ A better investment opportunity set will increase or decrease optimal consumption depending on the position of ψ with respect to one. If $\psi < 1$, optimal consumption will increase since the income effect dominates, while it will decrease for $\psi > 1$, since the investor will want to substitute consumption today for consumption tomorrow.¹¹

The infinite horizon problem reveals some constraints between the preference parameters. We show in Appendix B that the value function above is correct only if:¹²

$$\beta < r + \frac{q_S^2}{2\gamma\sigma_S^2} \quad \rightarrow \quad \text{Condition 1}, \quad (28)$$

which establishes a link between β and γ given a Sharpe ratio. This can be interpreted as follows: too impatient investors (who strongly discount future consumption) may not find optimal solutions if market conditions are poor, especially if they are very risk averse. Moreover, with an infinite horizon, the optimal consumption-to-wealth ratio will be constant and equal to:

$$\frac{C_t}{W_t} = \psi\beta + \left(r + \frac{q_S^2}{2\gamma\sigma_S^2}\right)(1 - \psi). \quad (29)$$

Non-negativity of consumption will therefore add a second condition, which can be interpreted as an upper bound on the elasticity of intertemporal substitution parameter given condition 1:

$$\psi < 1 + \frac{\beta}{r + \frac{q_S^2}{2\gamma\sigma_S^2} - \beta} \quad \rightarrow \quad \text{Condition 2}. \quad (30)$$

¹⁰Notice that this ratio remains positive for all parameter values, in particular ψ .

¹¹It is easy to see that the opportunity set is represented in the equations above by the term $\left(r + \frac{q_S^2}{2\gamma\sigma_S^2}\right)$, which is always multiplied by $(\psi - 1)$. Therefore, $\psi > 1$ or $\psi < 1$ will determine the final effect of a better investment set. Taking derivatives with respect to the opportunity set term is a way to determine which effect dominates in each case.

¹²Provided that $\gamma > 1$.

The condition above highlights that the propensity to postpone consumption (as measured by ψ) is limited from above, and this limit depends on the investor's impatience level and on market conditions (as represented by the riskless interest rate and the risky asset Sharpe ratio).

Going back to a finite horizon, we obtain that the optimal portfolio strategy is exactly the same as when $\psi = 1$, therefore constant and with no hedging demands:

$$\alpha_{S,t} = \frac{1}{\gamma} \frac{q_S}{\sigma_S^2}. \quad (31)$$

This result expands Svensson (1989), who finds that, for an investor with non-expected utility under a non-stochastic opportunity set and an infinite horizon, the optimal portfolio depends on the risk aversion parameter but not on the elasticity of intertemporal substitution.¹³ Another interesting result, also in Appendix B, is that if an investor has no motives for leaving a bequest, the necessity of consuming all her wealth will only be accomplished if ψ and γ are on opposite sides of one, meaning that no bequest will restrict ψ to be lower than one in the empirical relevant case (in which $\gamma > 1$).

3 Horizon and Preference Effects in a General Stochastic Setting

The aim of this section is to characterize and quantify the horizon effects and the role of each preference parameter in the optimal portfolio strategy. For this purpose, we generalize the previous setting by allowing the investor to also trade a bond and by making interest rates stochastic. We first describe this extended setting. We then propose an estimation strategy for the parameters of the model with a specific US data set. Before discussing the horizon and preference effects, we assess the accuracy of the approximation by comparing it to analytical solutions in restricted models.

3.1 The Economy

We assume that the spot interest rate follows the usual mean reverting Ornstein-Uhlenbeck process as in Vasicek (1977):

¹³Bhamra and Uppal (2006) also found the same result doing a simple three-date exercise under recursive preferences.

$$dr_t = \kappa_r (\bar{r} - r_t) dt + \sigma_r dZ_{r,t}, \quad (32)$$

where κ_r , \bar{r} and σ_r are, respectively, the speed of mean reversion, the long run mean and the volatility.

We also add a bond as a second risky asset to enrich the investor's menu. The bond return follows a similar process as the stock:

$$\frac{dB_t}{B_t} = (r_t + q_{B,t}) dt + \sigma_B dZ_{B,t}, \quad (33)$$

where σ_B denotes its volatility (assumed to be constant.¹⁴) The risk premium, $q_{B,t}$, is an affine function of the predictor as it was the case for the stock:

$$q_{B,t} = b_{B,0} + b_{B,1}P_t, \quad (34)$$

where $b_{B,j}$ is constant for $j = 0, 1$.

These processes for the interest rate and the bond return complete the model. Even though the interest rate represents an additional state variable, all results obtained in the simplified setting of the previous sections will carry through. We report here the approximate solutions for the optimal consumption and portfolio shares.

$$\frac{C_t}{W_t} = \beta^\psi \exp \left\{ (1 - \psi) \left[A_1(\tau) + A_2(\tau) r_t + A_3(\tau) P_t + \frac{A_4(\tau)}{2} P_t^2 \right] \right\} \quad (35a)$$

$$\begin{bmatrix} \alpha_{S,t} \\ \alpha_{B,t} \end{bmatrix} = \frac{1}{\gamma} \Sigma^{-1} \begin{bmatrix} b_{S,0} + b_{S,1}P_t \\ b_{B,0} + b_{B,1}P_t \end{bmatrix} + \frac{(1 - \gamma) A_2}{\gamma} \Sigma^{-1} \Sigma_r + \frac{(1 - \gamma) (A_3 + A_4 P_t)}{\gamma} \Sigma^{-1} \Sigma_p \quad (35b)$$

where $\Sigma = \begin{bmatrix} \sigma_S^2 & \rho_{SB} \sigma_S \sigma_B \\ \rho_{SB} \sigma_S \sigma_B & \sigma_B^2 \end{bmatrix}$ is the variance-covariance matrix of asset returns, $\Sigma_r = \begin{bmatrix} \rho_{S_r} \sigma_S \sigma_r \\ \rho_{B_r} \sigma_B \sigma_r \end{bmatrix}$ and $\Sigma_p = \begin{bmatrix} \rho_{S_p} \sigma_S \sigma_p \\ \rho_{B_p} \sigma_B \sigma_p \end{bmatrix}$.

The formulas are of course very similar to the ones described in the previous section with the simplified setting. The derivations of the set of equations to compute numerically the A_i coefficients are detailed in a technical appendix.¹⁵

¹⁴Usually the bond volatility is a function of the time to maturity of the bond but since we consider a constant maturity bond for simplicity, the constant volatility assumption is consistent with absence of arbitrage.

¹⁵This appendix is available online at <http://ssrn.com/abstract=2664173>.

3.2 Data and Estimated Parameters

To compute the optimal portfolio and consumption policies, we need to obtain estimates of the model parameters. As it can be seen in equation (35b) above, there are 18 parameters in total for the dynamics of the assets, the interest rate and the predictor: κ_r , \bar{r} , σ_r , κ_p , \bar{P} , σ_p , σ_S , σ_B , $b_{S,0}$, $b_{S,1}$, $b_{B,0}$, $b_{B,1}$ and all correlations between the relevant Brownian motions: ρ_{Sr} , ρ_{Br} , ρ_{SB} , ρ_{SP} , ρ_{BP} and ρ_{pr} . In Appendix E, we explain in detail the estimation procedure.

The short-term riskless rate (RF in Table 1) is the one-month US Treasury bill rate. For the stock (*Stock*) we chose value-weighted returns on the S&P500 index including dividends. For the bond (*Bond*) we selected the Bank of America Merrill Lynch US Corp & Govt Master bond index, downloaded from Datastream. For the predictor, we use the aggregate dividend yield (DY) on the S&P500 index, defined as the total dividend paid off during the current and last 11 months divided by the current value of the value-weighted market portfolio.

Table 1 about here

Panel A of Table 1 reports descriptive statistics for all time series. All returns are monthly, expressed in percentages and non-annualized. Bond returns are especially non-normal given the high-yield episode of the beginning of the eighties. Equity returns are negatively skewed while the risk-free rate and bond returns are positively skewed, a usual feature for these assets. We also find large first-order auto-regressive coefficients for the predictor and the interest rate, respectively 0.993 and 0.958, reflecting the high level of persistence in these variables.

Panel B reports the correlation matrix between all the variables. The risk-free interest rate is positively correlated with the predictor (0.68), but is practically uncorrelated with the risky assets. This does not preclude intensive intertemporal hedging activity since what matters for long horizon investors is not the instantaneous correlations but rather the correlation between stocks and bonds and the changes in the interest rate up to the investor's investment horizon. The latter may well be substantial due to the interaction of the changes in the market price of risk driven by the dividend yield which is highly correlated with the stock.

In Table 2 we report the estimated parameter values as well as the value assumed for the preference parameter β , which we will keep constant throughout our various scenarios in terms of horizon, risk aversion and attitude towards intertemporal substitution.

Table 2 about here

Because our data sample dates back to the 70s, the long-run mean for the interest rate is relatively high and equal to 0.42%, very close to the sample average of 0.44% reported in Panel A of Table 1. The volatility of the innovation is 0.08%, very low compared to the 0.27% volatility of the interest rate level. This is a direct consequence of the very high persistence of this variable. The mean reversion parameters κ_r is equal to 0.0434 implying an autoregressive coefficient equal to 0.957 ($\exp(-0.0434)$) very close to its corresponding discrete value 0.958. The same qualitative analysis holds also for the dividend yield.

Both the stock and the bond risk premia load positively on the dividend yield although the stock naturally loads more heavily on this variable. From the values reported in Table 2, we can compute the stock index and the bond index excess return long-run means, using equations (3) and (34) along with the long-run mean for the predictor: we find annualized values of 5.1% for the stock and 2.7% for the bond. The correlation between the stock and the bond is relatively low (0.1839) leaving room for potential gains from diversification. The stock's correlation with the dividend yield is very high (in absolute value) relative to the bond's one and, inversely, the bond's correlation with the interest rate is higher (in absolute value) than the stock's one.

Given these estimated parameter values, we can assess the quality of the approximate solution we proposed and investigate the horizon effects as well as the role of the EIS.

3.3 On the Accuracy of our Approximate Solution

We compare our approximation with the original log-linear approximation of Campbell et al. (2004) under infinite horizon. Their solution needed to be extended to our setting, which includes another risky asset and stochastic interest rate. Previewing our findings, it turns out that both approximations give very similar strategies. Table 3 presents some optimal choices as given by both solutions for some combinations of γ and ψ .

Table 3 about here

As it can be noted, both provide virtually the same choices. We also explored two cases where we can obtain exact solutions: the constrained case of Kraft et al.(2013) and the complete market setting of Wachter (2002). The respective exact solutions and our approximate ones are almost identical in both settings, for all configurations of parameters.¹⁶ Since both these papers

¹⁶See Figures 2 to 9 in our technical online appendix.

consider the case of finite horizon, they allow us to check that our approximate solution works for both long and short horizons.

The comparison suggests that our approximate solution is the natural extension of the log-linear approach to the finite horizon problem. In fact, the difference between our approximation around long-run state-variable values and the approximation around the long-run mean of consumption is due to the expected value of the term in P_t^2 (and obviously horizon effects since our approximation is built for finite horizons). This in turn brings about second-order effects which are of first importance for portfolio strategy.

3.4 Horizon Effects in Consumption and Optimal Portfolios

We show in Figure 1 the optimal consumption-to-wealth ratio as a function of the investment horizon for different values of the preference parameters.

Figure 1 about here

An important finding illustrated by this figure is the importance of the value of the EIS for the optimal consumption to wealth ratio. When $\psi = 1$, we already know that the optimal policy is kept constant and equal to β (0.17% monthly). When ψ is greater than 1, the investor has a high willingness to postpone consumption in the future, hence she adopts a very low policy: in this case, the horizon has almost no practical impact, since the investor prefers not to consume to better take advantage of the investment opportunities and leave a sizable bequest. However, when ψ is lower than one, the picture changes dramatically. First, the consumption to wealth ratio is much higher than in the previous case, the investor consuming a large fraction of her wealth at each period. Young people consume however less than old people, that is people very close to their investment horizon. For a one-year investment horizon, the ratio is close to 80% reflecting a very low desire to leave some wealth for bequest.

Another important implication of these findings is that infinite horizon effects are not all representative of the behavior of consumers with short investment horizons whenever ψ is lower than one while they are when ψ is greater than one. Hence the importance to look at consumption/investment issues for finite investment horizons.

We show in Figure 2 the optimal strategies associated with the optimal consumption-to-wealth ratios shown above. We present the total stock and bond asset allocations, but we also decompose these shares into the myopic part and intertemporal hedging demands with respect

to the interest rate and to the predictor. We consider an investor with $\gamma = 5$ or $\gamma = 20$ and $\psi = 0.75$ or $\psi = 1.25$. We assumed the predictor and the short term interest rate are at their long run mean (we offer a robustness check below).

Figure 2 about here

Figure 3 about here

Let us first consider the case $\psi = 0.75$. Recall that the consumption-to-wealth ratio was strongly impacted by the horizon effects. It turns out that the overall risky asset positions are also sensitive to the investment horizon. In our setting, the stock is strongly (negatively) correlated with the predictor relative to the bond, and the bond is highly (negatively) correlated with the spot rate relative to the stock. Since the correlation between the stock and the bond is relatively low, it is natural that the stock will attract much of the intertemporal hedging of the dividend yield while the bond will attract much of the hedging of the spot rate.

Several interesting conclusions could be drawn from Figure 2. First, the position in the bond is much higher than the corresponding one in the stock. This is a direct consequence of the low volatility of the bond relative to the stock. Second, for investment horizons greater than 5 years, the bond position is virtually flat while the stock's position smoothly increases with the investment horizon. When compared to the myopic component, horizon effects appear substantial both qualitatively and quantitatively. Overall, and consistently with the conclusion from the behavior of the consumption-to-wealth ratio, it appears important to take into account the investment horizon to determine the optimal portfolio positions under recursive utility. Our approximation provides an easy and accurate way to gauge these effects under various parameter configurations.

When $\psi = 1.25$, the picture is qualitatively similar to the previous case although now the portfolio strategy is more aggressive. The investor invests much more in the stock and this increase is directly related to the increase in the intertemporal hedging component relative to the dividend yield. Interestingly the bond position is relatively left unchanged. Recall that the investor allocates more wealth to the portfolio and less to consumption, but in relative terms the bond position is marginally affected compared to the substantial impact on the stock position.

While the role of the risk aversion is well understood, the role of the EIS is less obvious and some comparative statics may help assess the generality of the effects described above.

3.5 The Role of EIS

In the figures below we show the optimal consumption-to-wealth ratio as well as the optimal portfolio positions for different values of ψ .

Figure 4 about here

Figure 5 about here

Generalizing the previous results of Svensson (1989) (under infinite horizon) and Bhamra and Uppal (2006) (under a three-date exercise), we confirm that ψ will have no direct effect on the myopic part of the optimal portfolio. Its direct impact on the optimal portfolio will therefore be limited to hedging demands. Interestingly, the demands for the stock as well as for the bond are increasing in this parameter. The impact is more important for the stock than for the bond, and most importantly the horizon effects are reinforced in both cases. Looking at the behavior of the consumption-to-wealth ratio, the unitary value of the EIS is an important kink point. Below 1, the consumption-to-wealth ratio increases whenever the EIS decreases but less so when the investment horizon is long. In other words, the consumption-to-wealth ratio is more sensitive to the EIS for short-horizon investors than for long-horizon investors.

3.6 What If One Ignores the Intertemporal Hedging Components ?

In this sub-section, we assess the costs of ignoring intertemporal hedging demands. We consider an investor who follows a myopic portfolio strategy instead of the optimal one and compute the total relative utility loss that she incurs. To do so, we substitute in the Bellman equation the optimal consumption strategy and the myopic portfolio strategy given below:

$$C_t = \beta^\psi V_w^{-\psi} [(1 - \gamma) V_t]^{\frac{1-\psi\gamma}{1-\gamma}}, \quad (36a)$$

$$\begin{bmatrix} \alpha_{S,t,Myopic} \\ \alpha_{B,t,Myopic} \end{bmatrix} = \frac{V_w}{-W_t V_{ww}} \Sigma^{-1} \begin{bmatrix} \mathbf{b}_{S,0} + \mathbf{b}_{S,1} P_t \\ \mathbf{b}_{B,0} + \mathbf{b}_{B,1} P_t \end{bmatrix}. \quad (36b)$$

We follow exactly the same steps as in the optimal problem and the solution to this sub-optimal problem is given by the same value function as before, but with different coefficients.¹⁷ As this value function is sub-optimal, it will provide a utility value that is less than the one

¹⁷These coefficients solve a similar system of ordinary differential equations, which we show in the online appendix to this paper.

corresponding to the optimal value function. In Figure 6, we plot the utility loss in percentage (optimal value function over the myopic sub-optimal one minus one) as a function of the investor's horizon, for four different values of γ (we set $\psi = 0.75$ in the upper plot and $\psi = 1.25$ in the lower one).

Figure 6

The most notable feature is the huge importance, once again, of the value of the EIS. Whenever it is less than 1, the welfare loss increases smoothly with the investment horizon. For a forty-year investment horizon and $\gamma = 5$, the loss is 20% when $\psi = 0.75$. When $\psi = 1.25$, the welfare loss is remarkably greater than 60%. This is directly related to the importance of the portfolio decisions whenever $\psi = 1.25$ relative to consumption decisions. We stressed above that the value of this parameter impacts strongly the portfolio positions.

4 Conclusion

In this paper we have explored the importance of disentangling risk aversion from the elasticity of intertemporal substitution for an investor who has a finite horizon, in the presence of predictable time-varying investment opportunities and incomplete markets. We derived approximate analytical solutions for optimal consumption and portfolio decisions. In deriving this general result we developed exact analytical formulas that extend the literature for restricted environments such as a constant opportunity set or a unitary elasticity of intertemporal substitution.

In an application including stocks and bonds as risky assets, a stochastic interest rate and dividend yield as a predictor of returns, we show the quantitative importance of horizon effects and hedging demands on optimal decisions, as well as the sensitivity of the solutions to variations in risk aversion and the elasticity of intertemporal substitution. We show that optimal consumption decisions are mainly affected when ψ is less than one while optimal portfolios are quite sensitive to the EIS.

Under a constant opportunity setting, we show that the existence of solutions when letting the horizon go to infinity imposes two relevant restrictions on the preference parameters: These restrictions have never been seen before in the literature. Moreover, in the general case with no bequest, ψ and γ should be on opposite sides of one: this finding shall have important impact on empirically analyzed cases in which both parameters are set greater than one.

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Appendices

A Details of the Solution when $\psi = 1$ under a Constant Opportunity Set

In this case, the Bellman equation becomes:

$$0 = \beta (1 - \gamma) V_t \ln \left\{ \beta V_w^{-1} [V_t (1 - \gamma)]^{\frac{-\gamma}{1-\gamma}} \right\} - \frac{\partial V_t}{\partial \tau} + V_w W_t r - \beta (1 - \gamma) V_t - \frac{1}{2} \left(\frac{q_S}{\sigma_S} \right)^2 \frac{V_w^2}{V_{ww}}. \quad (37)$$

We guess that the value function will be $V(W_t, \tau) = h(\tau) \frac{W_t^{1-\gamma}}{1-\gamma}$ with terminal condition equal to $V(W_T, 0) = \epsilon \frac{W_T^{1-\gamma}}{1-\gamma}$, with $\epsilon = 0$ or 1 (a control to allow or not for utility from bequest), which implies $h(0) = \epsilon$. Substitution in equation (37) will cancel wealth out and imply an ordinary differential equation for the function $h(\tau)$:

$$0 = \beta \ln \beta - \frac{\beta}{1 - \gamma} \ln h - \frac{1}{1 - \gamma} \frac{dh}{d\tau} h^{-1} + r - \beta + \frac{1}{2\gamma} \left(\frac{q_S}{\sigma_S} \right)^2. \quad (38)$$

This equation has an exact solution given by:

$$h(\tau) = e^{(1-\gamma) \left(\ln \beta + \frac{r}{\beta} + \frac{q_S^2}{2\beta\gamma\sigma_S^2} - 1 - ke^{-\beta\tau} \right)}, \quad (39)$$

in which k is an integration constant. When we allow for utility from bequest, the terminal condition will imply $k = \ln \beta + \frac{r}{\beta} + \frac{q_S^2}{2\beta\gamma\sigma_S^2} - 1$. However, when felicity comes exclusively from consumption, the solution above will not verify the terminal condition. We need to impose $h(0) = 0$ since by equation (6), we need to have $V_T = 0$. Also, this condition forces the investor to consume everything when the horizon ends. The value function must incorporate this information. Indeed, one can see that the effect of wealth on the investor's felicity is proportional to $h(\tau)$: $\frac{\partial V_t}{\partial W_t} = h(\tau) W_t^{-\gamma}$. Assuming the terminal condition $h(0) = 0$ is a necessary condition to force the wealth effect on felicity to go towards zero. This implies that the investor will have to consume her entire wealth at maturity.¹⁸

Since equation (39) cannot verify the terminal condition for the problem without bequest, the value function cannot be of the form guessed. We then need another value function, but still such that both important conditions hold:

¹⁸In fact, after computing the optimal consumption strategy, we notice that this condition will indeed make the investor consume all her wealth.

I $V_T = 0$, due to the equation (6); and

II $\frac{\partial V_t}{\partial W_t}|_{\tau=0} = 0$ to guarantee that the investor will consume everything out of her wealth.

Our guess then is: $V(W_t, \tau) = h(\tau) \frac{W_t^{(1-\gamma)g(\tau)}}{1-\gamma}$. Observe that, from both conditions I and II above, the terminal condition $h(0) = 0$ is necessary. The terminal condition to g is also 0 because otherwise the investor would not consume everything at maturity.¹⁹ Substitution into the Bellman equation follows:

$$0 = \beta \ln \left(\beta g^{-1} h^{-\frac{1}{1-\gamma}} W_t^{1-g} \right) - \frac{1}{1-\gamma} \frac{dh}{d\tau} h^{-1} - \frac{dg}{d\tau} \ln W_t + gr - \beta - \frac{1}{2} \left(\frac{q_s}{\sigma_s} \right)^2 \frac{g}{[(1-\gamma)g-1]}. \quad (40)$$

Observe that the $\ln W_t$ term needs to cancel out because such equation cannot depend on wealth. This condition implies $\frac{dg(\tau)}{d\tau} = \beta [1 - g(\tau)]$, which, coupled with the terminal condition, gives $g(\tau) = 1 - e^{-\beta\tau}$. The ordinary differential equation for $h(\tau)$ becomes:

$$0 = \beta \ln \left(\frac{\beta}{1 - e^{-\beta\tau}} \right) - \frac{\beta}{1-\gamma} \ln h - \frac{1}{1-\gamma} \frac{dh}{d\tau} h^{-1} + (1 - e^{-\beta\tau}) r - \beta + \frac{1}{2} \left(\frac{q_s}{\sigma_s} \right)^2 \frac{1 - e^{-\beta\tau}}{\gamma + (1-\gamma)e^{-\beta\tau}}. \quad (41)$$

Under infinite horizon, the equation above shows that the function h becomes constant and equal to the previous function h (in the case of bequest and obviously also under infinite horizon). Therefore, it is easy to see that both value functions converge to the same one (along with all optimal choices): this is consistent because if horizon is infinite, allowing or not for bequest will make no difference at all for the investor.

For the finite horizon problem, we need only to guarantee that there exists a function $h(\tau)$ which solves (41) such that it admits $\text{Lim}_{\tau \rightarrow 0} h(\tau) = 0$ because this function will have absolutely no impact over the optimal strategies (this can be easily seen when substituting the value function guessed into both first-order conditions for optimality). The value function must also be thought as the limit $V_T(W_T, 0) = \text{Lim}_{\tau \rightarrow 0} V_t(W_t, \tau)$ because otherwise it will not be defined in the boundary. More precisely, the term $W_t^{(1-\gamma)(1-e^{-\beta\tau})}$ will not be defined at the boundary since we would have an undetermined form of the type 0^0 (both wealth and its exponent going towards zero). However, we have:

$$\text{Lim}_{\tau \rightarrow 0} W_t^{(1-\gamma)(1-e^{-\beta\tau})} = \text{Lim}_{\tau \rightarrow 0} (C_t \tau)^{(1-\gamma)(1-e^{-\beta\tau})} = \text{Lim}_{\tau \rightarrow 0} \tau^{(1-\gamma)(1-e^{-\beta\tau})} = 1, \quad (42)$$

where we have used the fact that $\text{Lim}_{\tau \rightarrow 0} \frac{C_t \tau}{W_t} = 1$.²⁰ Therefore, to have $\text{Lim}_{\tau \rightarrow 0} h(\tau) = 0$ is a sufficient condition to ensure that the value function will go towards zero when τ approaches

¹⁹To see this, just substitute our value function candidate into the first-order condition to consumption when $\psi = 1$ and apply the necessary condition to impose that the investor will consume all wealth: $\text{Lim}_{t \rightarrow T} \frac{C_t dt}{W_t} = 1$. This will imply that $g(0) = 0$.

²⁰To be rigorous, we had also used the fact that the instantaneous consumption rate process will not approach

zero.

Going back to equation (41), the function $h(\tau) = (1 - e^{-\beta\tau}) e^{f(\tau)}$ with $f(0) = 0$ will verify the desired terminal condition and will transform the ODE for $h(\tau)$ into the following:

$$\begin{aligned} \frac{df}{d\tau} = & \beta(\gamma - 2) \ln(1 - e^{-\beta\tau}) - \beta f - \frac{\beta e^{-\beta\tau}}{1 - e^{-\beta\tau}} + (1 - \gamma) (1 - e^{-\beta\tau}) r - \\ & -\beta(1 - \gamma)(1 - \ln \beta) + \frac{1}{2} \left(\frac{q_S}{\sigma_S} \right)^2 \frac{(1 - \gamma)(1 - e^{-\beta\tau})}{\gamma + (1 - \gamma)e^{-\beta\tau}}, \end{aligned} \quad (43)$$

which can be solved numerically. The optimal policies for both problems with and without bequest are then found substituting the respective value functions into the first-order conditions given by (9b) and (12).

B Details of the Solution when $\psi \neq 1$ under a Constant Opportunity Set

The Bellman equation becomes:

$$0 = \frac{\beta^\psi}{\psi - 1} V_w^{1-\psi} [(1 - \gamma) V_t]^{\frac{1-\psi\gamma}{1-\gamma}} - \frac{\psi\beta}{\psi - 1} (1 - \gamma) V_t - \frac{\partial V_t}{\partial \tau} + V_w W_t r - \frac{q_S^2}{2\sigma_S^2} \frac{V_w^2}{V_{ww}}. \quad (44)$$

We first show why a value function of the form $V(W_t, \tau) = h(\tau) \frac{W_t^{(1-\gamma)g(\tau)}}{1-\gamma}$ will not work if $\psi \neq 1$ (unless $g(\tau) = 1$). Substitution into equation (44) gives the following:

$$\begin{aligned} 0 = & \frac{1}{\psi - 1} \beta^\psi g^{1-\psi} h^{\frac{1-\psi}{1-\gamma}} W_t^{(1-\psi)(g-1)} - \frac{\psi}{\psi - 1} \beta - \\ & - \frac{1}{1 - \gamma} \frac{\partial h}{\partial \tau} h^{-1} - \frac{\partial g}{\partial \tau} \ln W_t + gr - \frac{q_S^2}{2\sigma_S^2} \frac{g}{[(1 - \gamma)g - 1]}, \end{aligned} \quad (45)$$

from which we clearly observe that this equation will only be independent of wealth (given that $\psi \neq 1$) if $g(\tau) = 1$. Therefore, our candidate guess remains $V(W_t, \tau) = h(\tau) \frac{W_t^{1-\gamma}}{1-\gamma}$ for both cases with or without bequest. The Bellman equation will then simplify to:

$$0 = \frac{1}{\psi - 1} \beta^\psi h^{\frac{1-\psi}{1-\gamma}} - \frac{1}{1 - \gamma} \frac{\partial h}{\partial \tau} h^{-1} - \frac{\psi}{\psi - 1} \beta + r + \frac{q_S^2}{2\gamma\sigma_S^2}. \quad (46)$$

zero as fast as time such that we will also have $\lim_{\tau \rightarrow 0} C_t^{(1-\gamma)(1-e^{-\beta\tau})} = 1$. This result follows when we couple the condition of consuming all wealth with the wealth dynamics.

The above ODE has the following exact analytical solution:

$$h(\tau) = \left\{ ke^{\left[\left(r + \frac{q_S^2}{2\gamma\sigma_S^2}\right)(\psi-1) - \psi\beta\right]\tau} - \frac{\beta^\psi}{\left(r + \frac{q_S^2}{2\gamma\sigma_S^2}\right)(\psi-1) - \psi\beta} \right\}^{\frac{1-\gamma}{\psi-1}}, \quad (47)$$

where k is a constant of integration. For the problem with bequest, the terminal condition will imply $k = 1 + \frac{\beta^\psi}{\left(r + \frac{q_S^2}{2\gamma\sigma_S^2}\right)(\psi-1) - \psi\beta}$. For the problem without bequest, the terminal condition should be $h(0) = 0$ because the value function when $\tau = 0$ must converge to zero. In such case, $k = \frac{\beta^\psi}{\left(r + \frac{q_S^2}{2\gamma\sigma_S^2}\right)(\psi-1) - \psi\beta}$. Optimal choices will result from the first-order conditions.

However, we must make sure that the value function for the finite horizon problem without bequest will satisfy conditions I and II discussed in the previous appendix. This needs to be carefully analyzed. The value function without bequest could be written in the following way:

$$V(W_t, \tau) = \left[\frac{\beta^\psi}{\left(r + \frac{q_S^2}{2\gamma\sigma_S^2}\right)(\psi-1) - \psi\beta} \right]^{\frac{1-\gamma}{\psi-1}} \left\{ e^{\left[\left(r + \frac{q_S^2}{2\gamma\sigma_S^2}\right)(\psi-1) - \psi\beta\right]\tau} - 1 \right\}^{\frac{1-\gamma}{\psi-1}} \frac{W_t^{1-\gamma}}{1-\gamma}. \quad (48)$$

We must then guarantee that $\lim_{\tau \rightarrow 0} V(W_t, \tau) = 0$: notice that in this case both conditions previously discussed will then be satisfied. When $\gamma > 1$, this limit is even more important because wealth approaches zero with a negative exponent. We can write this limit as follows:

$$\lim_{\tau \rightarrow 0} V(W_t, \tau) = \lim_{\tau \rightarrow 0} \left(\beta^\psi \tau \right)^{\frac{1-\gamma}{\psi-1}} \frac{W_t^{1-\gamma}}{1-\gamma}. \quad (49)$$

We can have the following cases:

- (a) $\gamma < 1$ and $\psi > 1 \implies$ The value function is zero, since we have no indetermination and the problem admits the exact solution given above;
- (b) $\gamma < 1$ and $\psi < 1 \implies$ The limit diverges and the value function will not verify the terminal condition, implying that the solution above is not adequate for the finite horizon problem without bequest;
- (c) $\gamma > 1$ and $\psi > 1 \implies$ The limit diverges and the value function will not verify the terminal condition, implying that the solution above is not adequate for the finite horizon problem without bequest; and
- (d) $\gamma > 1$ and $\psi < 1 \implies$ The limit converges to zero and the problem admits the exact solution

given above.

Given that γ has often been estimated at values greater than one, we would ultimately conclude that a solution to the investor's problem without bequest when $\psi > 1$ will be not guaranteed to exist.

B.1 Remarks on the Infinite Horizon Problem

We focus now on the infinite horizon problem with constant investment opportunities because it brings up interesting results. We can just take the limit when the horizon goes to infinity in the previous cases (with and without bequest) and see that both solutions will converge to the same results (as it should be). Otherwise, to be rigorous, we begin by the Bellman equation for this particular problem, but with the time-derivative term shut off:

$$0 = \frac{\beta^\psi}{\psi-1} V_w^{1-\psi} [(1-\gamma) V_t]^{\frac{1-\psi\gamma}{1-\gamma}} - \frac{\psi\beta}{\psi-1} (1-\gamma) V_t + V_w W_t r - \frac{q_S^2}{2\sigma_S^2} \frac{V_w^2}{V_{ww}}. \quad (50)$$

The value function that exactly solves this equation is given by (observe that the single state variable now is wealth):

$$V(W_t) = \left[\frac{\beta^\psi}{\psi\beta - \left(r + \frac{q_S^2}{2\gamma\sigma_S^2}\right) (\psi-1)} \right]^{\frac{1-\gamma}{\psi-1}} \frac{W_t^{1-\gamma}}{1-\gamma}. \quad (51)$$

The consumption-to-wealth ratio is then equal to:

$$\frac{C_t}{W_t} = \psi\beta - \left(r + \frac{q_S^2}{2\gamma\sigma_S^2}\right) (\psi-1). \quad (52)$$

As consumption cannot be negative (neither zero because all utility comes from consumption with infinite horizon), this suggests a natural upper bound on ψ , which is given by $1 + \frac{\beta}{r + \frac{q_S^2}{2\gamma\sigma_S^2} - \beta}$, provided that $r + \frac{q_S^2}{2\gamma\sigma_S^2} > \beta$.²¹

The optimal portfolio strategy is exactly the same as under finite horizon. Notice that these optimal choices coupled with wealth dynamics (5) will imply that:

$$\frac{dW_t}{W_t} = \left[\psi \left(r + \frac{q_S^2}{2\gamma\sigma_S^2} - \beta \right) + \frac{q_S^2}{2\gamma\sigma_S^2} \right] dt + \frac{q_S}{\gamma\sigma_S} dZ_{S,t}. \quad (53)$$

²¹This conclusion can also be found taking the infinite horizon case as the limit of the finite horizon problem.

Applying Ito's lemma and integrating, we find that:

$$W_{t+\Delta t} = W_t \exp \left\{ \left[\psi \left(r + \frac{q_S^2}{2\gamma\sigma_S^2} - \beta \right) + \frac{q_S^2}{2\gamma\sigma_S^2} - \frac{q_S^2}{2\gamma^2\sigma_S^2} \right] \Delta t + \frac{q_S}{\gamma\sigma_S} (Z_{S,t+\Delta t} - Z_{S,t}) \right\}, \quad (54)$$

from which we conclude that wealth is guaranteed to be always positive. On the other hand, we can substitute the value function into the aggregator (10a) and then use the original value function definition (6) to write the following:

$$\begin{aligned} V_t &= E_t \int_{u=t}^{\infty} f(C_u, J_u) du, \\ \left[\frac{\beta^\psi}{\psi \beta - \left(r + \frac{q_S^2}{2\gamma\sigma_S^2} \right) (\psi-1)} \right]^{\frac{1-\gamma}{\psi-1}} \frac{W_t^{1-\gamma}}{1-\gamma} &= \frac{1}{1-\frac{1}{\psi}} E_t \int_{u=t}^{\infty} \left\{ \beta^\psi V_w^{1-\psi} [(1-\gamma) V_t]^{\frac{1-\psi\gamma}{1-\gamma}} - \beta (1-\gamma) V_t \right\} du, \\ E_t \int_{u=t}^{\infty} W_u^{1-\gamma} du &= \frac{W_t^{1-\gamma}}{(1-\gamma)\psi \left(\beta - r - \frac{q_S^2}{2\gamma\sigma_S^2} \right)}. \end{aligned} \quad (55)$$

Since wealth is always positive, the expectation in the left-hand side of the equation above is surely positive, which leads us to conclude that the right-hand side must also be positive. Hence, for $\gamma > 1$, we end up with the necessary condition of $\beta < r + \frac{q_S^2}{2\gamma\sigma_S^2}$. This condition can be interpreted as an upper bound for β . If the rate of time preference parameter does not fulfill such condition, then the infinite horizon problem does not admit the solution discussed.

C Details of the Solution when $\psi = 1$ under a Stochastic Opportunity Set with Bequest

Based on the results under a constant opportunity set, we guess that the value function that solves the finite horizon problem with bequest is:

$$V(W_t, P_t, \tau) = I(P_t, \tau) \frac{W_t^{1-\gamma}}{1-\gamma}, \quad (56)$$

in which terminal conditions imply $I(P_t, 0) = 1$. The resulting differential equation for I follows from substituting the guess and $\psi = 1$ into the Bellman equation (13):

$$\begin{aligned} 0 &= r - \beta (1 - \ln \beta) - \frac{\beta}{1-\gamma} \ln I - \frac{1}{1-\gamma} \frac{\partial I}{\partial \tau} I^{-1} + \frac{1}{1-\gamma} \frac{\partial I}{\partial P} I^{-1} \kappa_p (\bar{P} - P_t) + \\ &+ \frac{1}{2} \frac{1}{1-\gamma} \frac{\partial^2 I}{\partial P^2} I^{-1} \sigma_p^2 + \frac{1}{2\gamma} \left(\frac{b_{S,0} + b_{S,1} P_t}{\sigma_S} \right)^2 + \frac{1}{\gamma} \frac{\partial I}{\partial P} I^{-1} \frac{b_{S,0} + b_{S,1} P_t}{\sigma_S} \rho_{SP} \sigma_p + \\ &+ \frac{1}{2\gamma} \left(\frac{\partial I}{\partial P} \right)^2 I^{-2} \rho_{SP}^2 \sigma_p^2. \end{aligned} \quad (57)$$

To complete the solution, we guess that:

$$I(P_t, \tau) = \exp \left\{ (1 - \gamma) \left[A_1(\tau) + A_2(\tau) P_t + \frac{A_3(\tau)}{2} P_t^2 \right] \right\}, \quad (58)$$

in which $A_i(\tau)$, $i = 1, 2, 3$, are functions of time left to the investment horizon ($\tau = T - t$). These coefficients depend on the primitive parameters of the model describing investment opportunities and preferences. To find boundary conditions, we first recall that the terminal value function cannot depend on any state variable, which lead us to $A_i(0) = 0$, $i = 2, 3$. The boundary condition to $A_1(\tau)$ is also $A_1(0) = 0$ because $V(W_T, P_T, 0) = \frac{W_T^{1-\gamma}}{1-\gamma}$.

We can substitute expression (58) into equation (57) (for clarity, we suppress the time dependence subscript in functions $A_i(\tau)$):

$$\begin{aligned} 0 = & r - \beta(1 - \ln \beta) - \beta \left(A_1 + A_2 P_t + \frac{A_3}{2} P_t^2 \right) - \left(A_1' + A_2' P_t + \frac{A_3'}{2} P_t^2 \right) + \\ & + (A_2 + A_3 P_t) \kappa_p (\bar{P} - P_t) + \frac{1}{2} \left[A_3 + (1 - \gamma) (A_2 + A_3 P_t)^2 \right] \sigma_p^2 + \\ & + \frac{1}{2\gamma} \left(\frac{b_{s,0} + b_{s,1} P_t}{\sigma_s} \right)^2 + \frac{1-\gamma}{\gamma} (A_2 + A_3 P_t) \frac{b_{s,0} + b_{s,1} P_t}{\sigma_s} \rho_{SP} \sigma_p + \\ & + \frac{(1-\gamma)^2}{2\gamma} (A_2 + A_3 P_t)^2 \rho_{SP}^2 \sigma_p^2. \end{aligned} \quad (59)$$

Collecting terms, the coefficients $A_i(\tau)$, $i = 1, \dots, 3$ obey a system of ordinary differential equations with boundary conditions $A_i(0) = 0$, $i = 1, \dots, 3$ that we solve numerically.²²

Equation a (independent term):

$$\begin{aligned} 0 = & r + \beta(\ln \beta - 1) - \beta A_1 - A_1' + A_2 \kappa_p \bar{P} + \frac{1}{2} \left[A_3 + (1 - \gamma) A_2^2 \right] \sigma_p^2 + \\ & + \frac{1}{2\gamma} \frac{b_{s,0}^2}{\sigma_s^2} + \frac{1-\gamma}{\gamma} A_2 \frac{b_{s,0}}{\sigma_s} \rho_{SP} \sigma_p + \frac{(1-\gamma)^2}{2\gamma} A_2^2 \rho_{SP}^2 \sigma_p^2. \end{aligned} \quad (60a)$$

Equation b (P_t term):

$$\begin{aligned} 0 = & -\beta A_2 - A_2' - A_2 \kappa_p + A_3 \kappa_p \bar{P} + (1 - \gamma) A_2 A_3 \sigma_p^2 + \\ & + \frac{1}{\gamma} \frac{b_{s,0} b_{s,1}}{\sigma_s^2} + \frac{1-\gamma}{\gamma} \left(A_3 \frac{b_{s,0}}{\sigma_s} + A_2 \frac{b_{s,1}}{\sigma_s} \right) \rho_{SP} \sigma_p + \frac{(1-\gamma)^2}{\gamma} A_2 A_3 \rho_{SP}^2 \sigma_p^2. \end{aligned} \quad (60b)$$

Equation c (P_t^2 term):

$$\begin{aligned} 0 = & -\beta \frac{A_3}{2} - \frac{A_3'}{2} - A_3 \kappa_p + \frac{1}{2} (1 - \gamma) A_3^2 \sigma_p^2 + \frac{1}{2\gamma} \frac{b_{s,1}^2}{\sigma_s^2} + \\ & + \frac{1-\gamma}{\gamma} A_3 \frac{b_{s,1}}{\sigma_s} \rho_{SP} \sigma_p + \frac{(1-\gamma)^2}{2\gamma} A_3^2 \rho_{SP}^2 \sigma_p^2. \end{aligned} \quad (60c)$$

²²This procedure implies that all terms on the state variables (powers and cross-products too) are equal to zero. The reason is that the PDE equation must be equal to zero independently of the current state variable values.

D Details of the Approximate Solution when $\psi \neq 1$ under a Stochastic Opportunity Set with Bequest

Substituting the approximate expression for the log consumption-to-wealth ratio in (19) into (17), we obtain:

$$0 = \frac{1}{\psi-1} \beta^\psi \exp \left[(1-\psi) \left(A_1 + A_2 \bar{P} + \frac{A_3}{2} \bar{P}^2 \right) \right] * \left\{ 1 + (1-\psi) \left[A_2 (P_t - \bar{P}) + \frac{A_3}{2} (P_t^2 - \bar{P}^2) \right] \right\} - \frac{\beta^\psi}{\psi-1} + r - \left(A'_1 + A'_2 P_t + \frac{A'_3}{2} P_t^2 \right) + (A_2 + A_3 P_t) \kappa_p (\bar{P} - P_t) + \frac{1}{2} \left[A_3 + (1-\gamma) (A_2 + A_3 P_t)^2 \right] \sigma_p^2 + \frac{1}{2\gamma} \left(\frac{b_{s,0} + b_{s,1} P_t}{\sigma_s} \right)^2 + \frac{1-\gamma}{\gamma} (A_2 + A_3 P_t) \frac{b_{s,0} + b_{s,1} P_t}{\sigma_s} \rho_{SP} \sigma_p + \frac{(1-\gamma)^2}{2\gamma} (A_2 + A_3 P_t)^2 \rho_{SP}^2 \sigma_p^2. \quad (61)$$

To obtain $A_i(\tau)$, $i = 1, 2, 3$, we solve numerically the system below of ordinary differential equations.

Equation a (independent term):

$$0 = \frac{1}{\psi-1} \beta^\psi \exp \left[(1-\psi) \left(A_1 + A_2 \bar{P} + \frac{A_3}{2} \bar{P}^2 \right) \right] * \left[1 - (1-\psi) \left(A_2 \bar{P} + \frac{A_3}{2} \bar{P}^2 \right) \right] - \frac{\beta^\psi}{\psi-1} + r - A'_1 + A_2 \kappa_p \bar{P} + \frac{1}{2} \left[A_3 + (1-\gamma) A_2^2 \right] \sigma_p^2 + \frac{1}{2\gamma} \left(\frac{b_{s,0}}{\sigma_s} \right)^2 + \frac{1-\gamma}{\gamma} A_2 \frac{b_{s,0}}{\sigma_s} \rho_{SP} \sigma_p + \frac{(1-\gamma)^2}{2\gamma} A_2^2 \rho_{SP}^2 \sigma_p^2. \quad (62a)$$

Equation b (P_t term):

$$0 = -\beta^\psi A_2 \exp \left[(1-\psi) \left(A_1 + A_2 \bar{P} + \frac{A_3}{2} \bar{P}^2 \right) \right] - A'_2 - A_2 \kappa_p + A_3 \kappa_p \bar{P} + (1-\gamma) A_2 A_3 \sigma_p^2 + \frac{1}{\gamma} \frac{b_{s,0} b_{s,1}}{\sigma_s^2} + \frac{1-\gamma}{\gamma} \left(A_2 \frac{b_{s,1}}{\sigma_s} + A_3 \frac{b_{s,0}}{\sigma_s} \right) \rho_{SP} \sigma_p + \frac{(1-\gamma)^2}{\gamma} A_2 A_3 \rho_{SP}^2 \sigma_p^2. \quad (62b)$$

Equation c (P_t^2 term):

$$0 = -\beta^\psi \frac{A_3}{2} \exp \left[(1-\psi) \left(A_1 + A_2 \bar{P} + \frac{A_3}{2} \bar{P}^2 \right) \right] - \frac{A'_3}{2} - A_3 \kappa_p + \frac{1}{2} (1-\gamma) A_3^2 \sigma_p^2 + \frac{1}{2\gamma} \frac{b_{s,1}^2}{\sigma_s^2} + \frac{1-\gamma}{\gamma} A_3 \frac{b_{s,1}}{\sigma_s} \rho_{SP} \sigma_p + \frac{(1-\gamma)^2}{2\gamma} A_3^2 \rho_{SP}^2 \sigma_p^2. \quad (62c)$$

E Estimation Procedure

Since our model is written in continuous time and we observe discrete-time data, we need to discretize each process. For Ornstein-Uhlenbeck processes, we can obtain closed-form analytical solutions, which allow us to find consistent estimators for the continuous-time parameters based on discrete data. For the interest rate process given by (32), we integrate the process $d(r_t e^{\kappa r t})$ from t to $t+1$ (one-month time step) to obtain the following:

$$r_{t+1} = \bar{r} (1 - e^{-\kappa r}) + e^{-\kappa r} r_t + \sigma_r e^{-\kappa r} \int_t^{t+1} e^{\kappa r (u-t)} dZ_{r,u}. \quad (63)$$

We then run a first-order auto-regression for the discretized observations of r_t using market data:

$$r_{t+1} = a_r + A_r r_t + v_{r,t+1}. \quad (64)$$

and identify terms in equations (63) and (64) to estimate the parameters:

$$\kappa_r = -\ln A_r, \quad (65a)$$

$$\bar{r} = \frac{a_r}{1 - A_r}, \quad (65b)$$

$$\sigma_r^2 = \frac{2 \ln A_r}{A_r^2 - 1} \sigma_{v_r}^2. \quad (65c)$$

We proceed in exactly the same way with the predictor, running an analogous first-order autoregressive regression and comparing it with the integral solution to its process to find the analogous equations below:

$$\kappa_p = -\ln A_p, \quad (66a)$$

$$\bar{p} = \frac{a_p}{1 - A_p}, \quad (66b)$$

$$\sigma_p^2 = \frac{2 \ln A_p}{A_p^2 - 1} \sigma_{v_p}^2. \quad (66c)$$

We can compute the covariance between the discretized terms P_{t+1} and r_{t+1} as below:

$$\text{Cov}_t(P_{t+1}, r_{t+1}) = \text{Cov}_t(v_{p,t+1}, v_{r,t+1}) = \frac{\rho_{pr} \sigma_p \sigma_r}{\kappa_p + \kappa_r} (1 - e^{-\kappa_p - \kappa_r}), \quad (67a)$$

such that we now solve for the continuous-time correlation between the interest rate and the predictor:

$$\rho_{pr} = \text{Cov}_t(v_{p,t+1}, v_{r,t+1}) \frac{\kappa_p + \kappa_r}{\sigma_p \sigma_r} (1 - e^{-\kappa_p - \kappa_r})^{-1}. \quad (67b)$$

For the stock asset, we apply Ito's lemma to $\ln S_t$, using both equations (2) and (3):

$$\ln S_{t+1} - \ln S_t = \left(b_{S,0} - \frac{\sigma_S^2}{2} \right) + \int_t^{t+1} r_u du + b_{S,1} \int_t^{t+1} P_u du + \sigma_S \int_t^{t+1} dZ_{S,u}. \quad (68)$$

Because P_t follows an Ornstein-Uhlenbeck process, we have that:

$$\int_t^{t+1} P_u du = \left(1 - \frac{1 - e^{-\kappa_p}}{\kappa_p} \right) \bar{p} + \frac{1 - e^{-\kappa_p}}{\kappa_p} P_t + \frac{\sigma_p}{\kappa_p} \int_t^{t+1} \left[1 - e^{-\kappa_p(t+1-u)} \right] dZ_{P,u}, \quad (69)$$

which allows us to write:

$$\begin{aligned} \ln \frac{S_{t+1}}{S_t} - \int_t^{t+1} r_u du &= \left(b_{S,0} - \frac{\sigma_S^2}{2} \right) + b_{S,1} \left(1 - \frac{1 - e^{-\kappa_p}}{\kappa_p} \right) \bar{P} + b_{S,1} \frac{1 - e^{-\kappa_p}}{\kappa_p} P_t \\ &+ b_{S,1} \frac{\sigma_p}{\kappa_p} \int_t^{t+1} \left[1 - e^{-\kappa_p(t+1-u)} \right] dZ_{P,u} + \sigma_S \int_t^{t+1} dZ_{S,u}. \end{aligned} \quad (70)$$

A natural regression for identifying the parameters is then:

$$\ln \frac{S_{t+1}}{S_t} - \int_t^{t+1} r_u du = a_S + A_S P_t + v_{S,t+1}, \quad (71)$$

where $\ln \frac{S_{t+1}}{S_t}$ is the continuously compounded period total return on the stock asset between dates t and $t + 1$.²³ Identifying equivalent terms in (70) and in (71) yields:

$$b_{S,1} = \frac{\kappa_p}{1 - e^{-\kappa_p}} A_S, \quad (72a)$$

$$b_{S,0} = a_S + \frac{\sigma_S^2}{2} - b_{S,1} \left(1 - \frac{1 - e^{-\kappa_p}}{\kappa_p} \right) \bar{P}. \quad (72b)$$

Considering the discretized forms of the processes above, we can write the following equations for the stock variance and covariances:

$$\begin{aligned} \sigma_{v_{S,t+1}}^2 &= \sigma_S^2 \int_t^{t+1} du + \left(\frac{\sigma_p}{\kappa_p} b_{S,1} \right)^2 \int_t^{t+1} \left[1 - e^{-\kappa_p(t+1-u)} \right]^2 du + \\ &+ 2 \frac{\sigma_S \sigma_p \rho_{SP}}{\kappa_p} b_{S,1} \int_t^{t+1} \left[1 - e^{-\kappa_p(t+1-u)} \right] du, \end{aligned} \quad (73a)$$

$$\begin{aligned} \text{Cov}_t(v_{S,t+1}, v_{r,t+1}) &= \sigma_S \sigma_r \rho_{Sr} e^{-\kappa_r(t+1)} \int_t^{t+1} e^{\kappa_r u} du + \\ &+ \frac{\sigma_p \sigma_r \rho_{pr}}{\kappa_p} b_{S,1} e^{-\kappa_r(t+1)} \int_t^{t+1} \left[1 - e^{-\kappa_p(t+1-u)} \right] e^{\kappa_r u} du, \end{aligned} \quad (73b)$$

$$\begin{aligned} \text{Cov}_t(v_{S,t+1}, v_{p,t+1}) &= \sigma_S \sigma_p \rho_{SP} e^{-\kappa_p(t+1)} \int_t^{t+1} e^{\kappa_p u} du + \\ &+ \frac{\sigma_p^2}{\kappa_p} b_{S,1} e^{-\kappa_p(t+1)} \int_t^{t+1} \left[1 - e^{-\kappa_p(t+1-u)} \right] e^{\kappa_p u} du. \end{aligned} \quad (73c)$$

²³It is important to note that in all these discrete-time computations, r_t should be also the continuously compounded interest rate for the period beginning at time t and finishing at time $t + 1$. Therefore, when going to the data, it is crucial to correctly identify whether the interest rate series refers to the next period free-rate or to the *realized* free rate. An incorrect identification leads to fairly wrong results!

Solving this system in a convenient order yields:

$$\sigma_S \sigma_r \rho_{Sr} = \frac{\kappa_r}{1 - e^{-\kappa_r}} \text{Cov}_t(v_{S,t+1}, v_{r,t+1}) - \frac{\sigma_p \sigma_r \rho_{pr}}{\kappa_p} b_{S,1} \left(1 - \frac{\kappa_r}{\kappa_r + \kappa_p} \frac{1 - e^{-\kappa_r - \kappa_p}}{1 - e^{-\kappa_r}} \right), \quad (74a)$$

$$\sigma_S \sigma_p \rho_{SP} = \frac{\kappa_p}{1 - e^{-\kappa_p}} \text{Cov}_t(v_{S,t+1}, v_{p,t+1}) - \frac{\sigma_p^2}{\kappa_p} b_{S,1} \left(\frac{1 - e^{-\kappa_p}}{2} \right), \quad (74b)$$

$$\sigma_S^2 = \sigma_{v_{S,t+1}}^2 - 2 \frac{\sigma_S \sigma_p \rho_{SP}}{\kappa_p} b_{S,1} \left(1 - \frac{1 - e^{-\kappa_p}}{\kappa_p} \right) - \left(\frac{\sigma_p}{\kappa_p} b_{S,1} \right)^2 \left(1 - 2 \frac{1 - e^{-\kappa_p}}{\kappa_p} + \frac{1 - e^{-2\kappa_p}}{2\kappa_p} \right). \quad (74c)$$

Finally, considering the second risky asset, the approach follows exactly the one we have just developed for the stock asset, which allows us to write the same equations (70) and (71) for the bond asset (just replacing S for B). Therefore, we will have the following equations for the bond parameters (the same structure as the one for the stock asset):

$$b_{B,1} = \frac{\kappa_p}{1 - e^{-\kappa_p}} A_B, \quad (75a)$$

$$b_{B,0} = a_B + \frac{\sigma_B^2}{2} - b_{B,1} \left(1 - \frac{1 - e^{-\kappa_p}}{\kappa_p} \right) \bar{P}, \quad (75b)$$

$$\sigma_B \sigma_r \rho_{Br} = \frac{\kappa_r}{1 - e^{-\kappa_r}} \text{Cov}_t(v_{B,t+1}, v_{r,t+1}) - \frac{\sigma_p \sigma_r \rho_{pr}}{\kappa_p} b_{B,1} \left(1 - \frac{\kappa_r}{\kappa_r + \kappa_p} \frac{1 - e^{-\kappa_r - \kappa_p}}{1 - e^{-\kappa_r}} \right), \quad (75c)$$

$$\sigma_B \sigma_p \rho_{BP} = \frac{\kappa_p}{1 - e^{-\kappa_p}} \text{Cov}_t(v_{B,t+1}, v_{p,t+1}) - \frac{\sigma_p^2}{\kappa_p} b_{B,1} \left(\frac{1 - e^{-\kappa_p}}{2} \right), \quad (75d)$$

$$\sigma_B^2 = \sigma_{v_{B,t+1}}^2 - 2 \frac{\sigma_B \sigma_p \rho_{BP}}{\kappa_p} b_{B,1} \left(1 - \frac{1 - e^{-\kappa_p}}{\kappa_p} \right) - \left(\frac{\sigma_p}{\kappa_p} b_{B,1} \right)^2 \left(1 - 2 \frac{1 - e^{-\kappa_p}}{\kappa_p} + \frac{1 - e^{-2\kappa_p}}{2\kappa_p} \right). \quad (75e)$$

The last parameter to be determined is the correlation between the two risky assets, which we obtain by comparing equations (70), (71) and the respective ones for the bond asset. We are then able to write the following:

$$\begin{aligned} \sigma_S \sigma_B \rho_{SB} = & \text{Cov}_t(v_{S,t+1}, v_{B,t+1}) - b_{S,1} b_{B,1} \left(\frac{\sigma_p}{\kappa_p} \right)^2 \left(1 - 2 \frac{1 - e^{-\kappa_p}}{\kappa_p} + \frac{1 - e^{-2\kappa_p}}{2\kappa_p} \right) - \\ & - \left(\frac{\sigma_B \sigma_p \rho_{BP}}{\kappa_p} b_{S,1} + \frac{\sigma_S \sigma_p \rho_{SP}}{\kappa_p} b_{B,1} \right) \left(1 - \frac{1 - e^{-\kappa_p}}{\kappa_p} \right). \end{aligned} \quad (76)$$

Table 1: We report in Panel A descriptive statistics for the series we use. All returns are monthly (non-annualized). The risk-free interest rate is continuously compounded. Panel B presents the correlation matrix for the time series. Our data set encompasses 418 monthly observations, from March 1976 till December 2010.

Panel A	Mean	Std Error	Std Dev	Skewness	Kurtosis	AR(1) Slope	Std Error
RF	0.44%	0.01%	0.27%	0.77	3.99	0.958	0.015
DY	2.75%	0.06%	1.13%	0.28	1.83	0.993	0.006
Stock	0.97%	0.22%	4.42%	-0.61	5.05	0.047	0.049
Bond	0.69%	0.09%	1.82%	1.80	16.14	0.020	0.049
Panel B	RF	DY	Stock	Bond			
RF	1.00	0.68	0.03	-0.01			
DY	0.68	1.00	-0.02	0.09			
Stock	0.03	-0.02	1.00	0.19			
Bond	-0.01	0.09	0.19	1.00			

Table 2: Parameter estimates used in this paper. All of them represent monthly values for the continuous-time processes.

$\beta = 0.0017$	$\kappa_r = 0.0434$	$\bar{r} = 0.0042$	$\sigma_r = 0.0008$	$\kappa_p = 0.0075$
$\bar{P} = 0.0222$	$\sigma_p = 0.0014$	$\sigma_S = 0.0447$	$\sigma_B = 0.0177$	$b_{S,0} = -0.0003$
$b_{S,1} = 0.2006$	$b_{B,0} = 0.0021$	$b_{B,1} = 0.0079$	$\rho_{Sr} = 0.0617$	$\rho_{Br} = -0.3784$
$\rho_{SB} = 0.1839$	$\rho_{Sp} = -0.8758$	$\rho_{BP} = -0.2287$	$\rho_{pr} = -0.0419$	

Table 3: We show below the optimal choices as given by the original log-linear solution and the corresponding values from our approximate solution for the infinite horizon problem and different values for ψ and γ . Our approximate choices are written with hats. All the values are in percentage.

$\psi = 0.75$	$\alpha_{S,t}$	$\hat{\alpha}_{S,t}$	$\alpha_{B,t}$	$\hat{\alpha}_{B,t}$	$\frac{C_t}{W_t}$	$\frac{\hat{C}_t}{\hat{W}_t}$
$\gamma = 5$	59.53	59.63	168.34	168.36	0.32	0.32
$\gamma = 10$	33.04	33.08	106.12	106.14	0.27	0.27
$\gamma = 20$	15.62	15.63	74.48	74.48	0.21	0.21
$\gamma = 40$	5.23	5.23	58.89	58.89	0.12	0.12
$\psi = 1.25$	$\alpha_{S,t}$	$\hat{\alpha}_{S,t}$	$\alpha_{B,t}$	$\hat{\alpha}_{B,t}$	$\frac{C_t}{W_t}$	$\frac{\hat{C}_t}{\hat{W}_t}$
$\gamma = 5$	84.37	84.14	175.89	175.83	0.005	0.006
$\gamma = 10$	44.61	44.60	110.31	110.31	0.06	0.06
$\gamma = 20$	18.28	18.27	75.82	75.81	0.12	0.12
$\gamma = 40$	4.02	4.02	57.84	57.84	0.21	0.21

Figure 1: Horizon effects on the optimal consumption strategy for different values of γ and ψ .

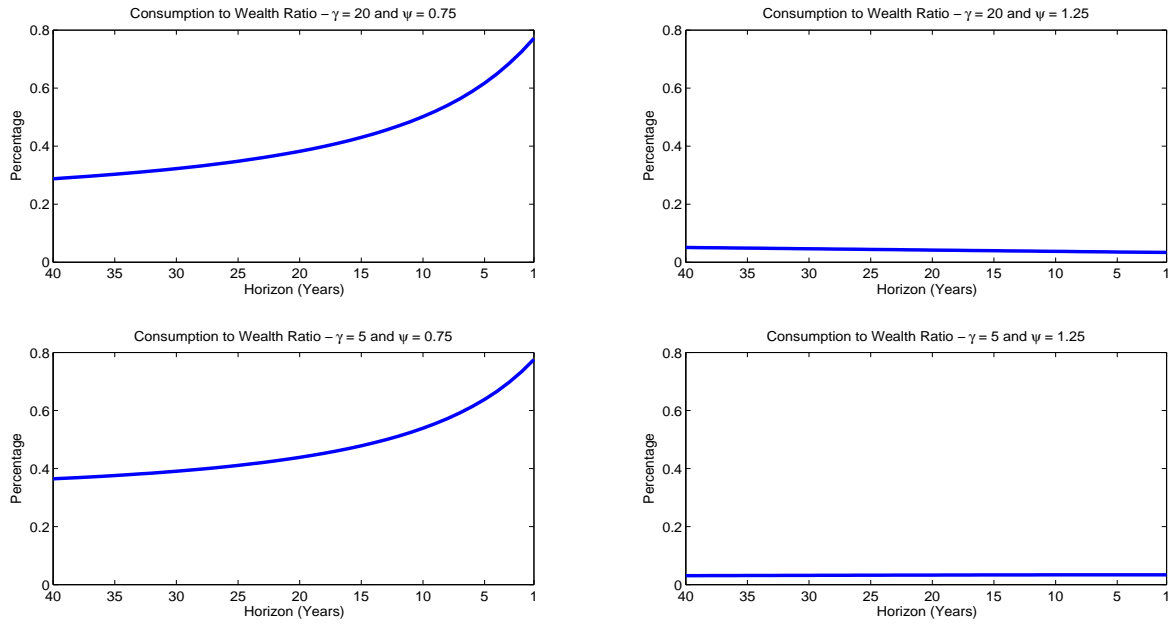


Figure 2: Horizon effects on the optimal portfolio strategies. We plot below the total allocations for an horizon of up to 40 years for an investor with $\psi = 0.75$. We also show the myopic part and the hedging demands of the portfolio. We have two hedging terms, one for each state variable, which are the short-term interest rate and the predictor. The upper plots are for an investor with $\gamma = 20$, while the lower plots refer to an investor with $\gamma = 5$.

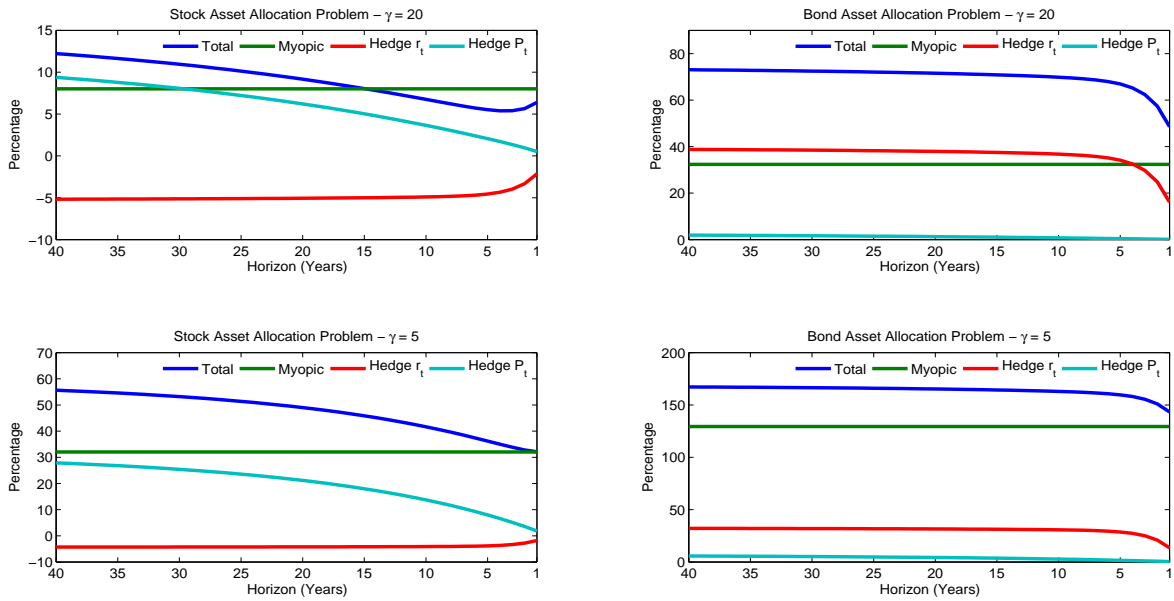


Figure 3: Horizon effects on the optimal portfolio strategies. We plot below the total allocations for an horizon of up to 40 years for an investor with $\psi = 1.25$. We also show the myopic part and the hedging demands of the portfolio. We have two hedging terms, one for each state variable, which are the short-term interest rate and the predictor. The upper plots are for an investor with $\gamma = 20$, while the lower plots refer to an investor with $\gamma = 5$.

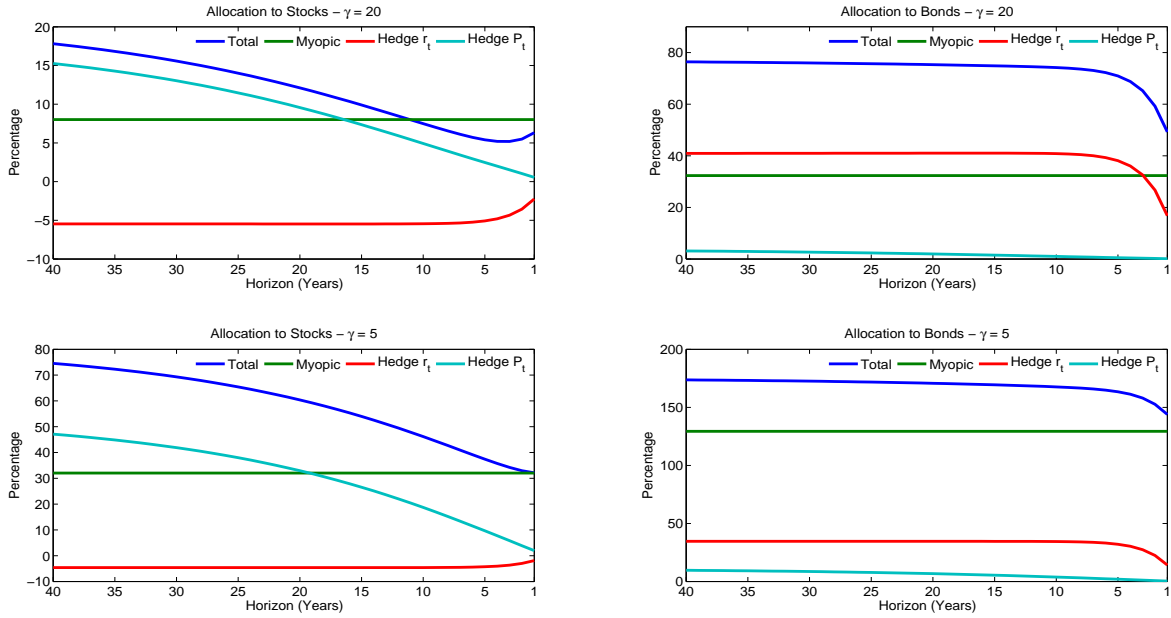


Figure 4: The effect of ψ on the optimal portfolio for different horizons. The stock asset allocations are plotted on the left panel, the bond asset allocations on the right panel. The upper plots have $\gamma = 20$, while the lower plots have $\gamma = 5$. The horizon T is given in years.

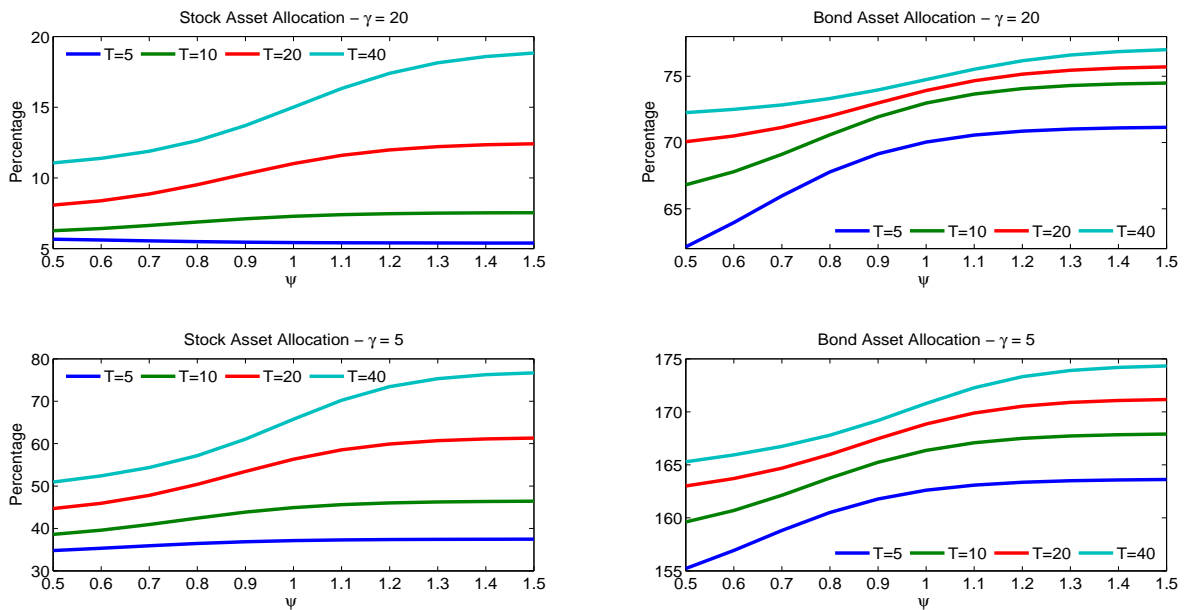


Figure 5: The effect of ψ on the optimal consumption-to-wealth instantaneous rate for different horizons. The upper plot has $\gamma = 20$, while the lower plot has $\gamma = 5$. The horizon T is given in years.

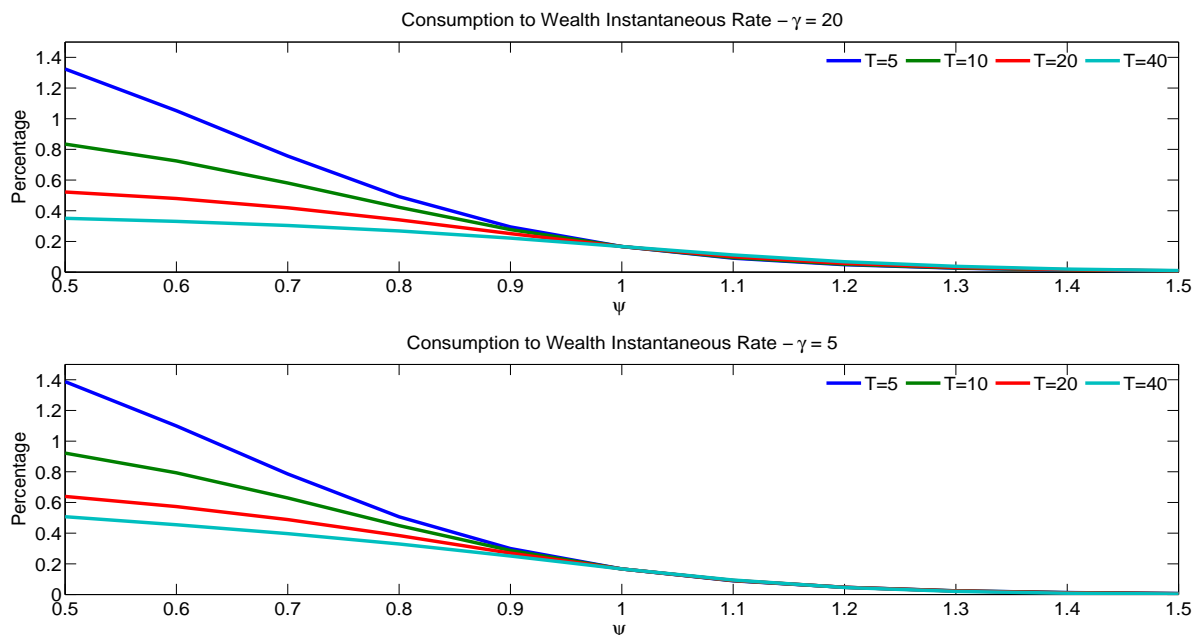


Figure 6: We plot below the percentage utility loss due to a myopic portfolio strategy. We compare the optimal value function with the sub-optimal one of an investor who follows a portfolio strategy without both hedging demand terms. We present the curves for four different risk-averse investors. The upper plot is for an investor with $\psi = 0.75$, while the lower one refers to an investor with $\psi = 1.25$.

