Optimal Portfolio Strategies in the Presence of Regimes in Asset Returns*

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Abstract

This paper analyzes an investor's optimal strategies in a regime switching economy with unobservable states and predictability of risky asset returns. We develop approximate analytical solutions to the unconstrained dynamic problem. The approximation is shown to be accurate in a simple two-regime setting. While the portfolio policy depends strongly on the current state of the economy, the consumption-to-wealth ratio is roughly state-independent. Predictability changes considerably the optimal portfolios. Hedging demands are negligible with regimes and no predictability, but can be important with predictability. On the other hand, the consumption-to-wealth ratio is not very impacted by the predictor.

JEL classification: G11, C02

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*This paper includes an online appendix in which we provide detailed derivations and extensions at `http://ssrn.com/abstract=BLABLABLA`.

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1 Introduction

Since the seminal paper of Hamilton (1989), Markov switching models have been widely used in diverse fields of finance (see a survey by Ang and Timmermann (2012)). In asset allocation, regime switching models of asset returns are able, for example, to capture a higher correlation structure in bear periods or heteroscedasticity through distinct volatilities in different regimes. Our interest is to apply this class of models in a general investor's problem with finite horizon, in which the agent optimally chooses portfolio allocation and (possibly) consumption with stochastic differential utility preferences.

Several papers have studied optimal portfolio problems holdings under regime-switching processes in discrete time. Ang and Bekaert (2002) solved numerically a two-regime portfolio problem with international assets, while Guidolin and Timmermann (2007) use Monte Carlo techniques to solve an allocation between large and small stocks and bonds in a four-state regime switching model. In continuous time, Liu (2011) examines a consumption-portfolio problem in which the expected return of a single risky asset follows a hidden Markov chain, under ambiguity. The investor's optimal choices are not in closed-form, but characterized in terms of Malliavin derivatives and stochastic integrals[1].

To the best of our knowledge, only two papers obtain explicit solutions to the investor problem in financial markets with regime switching. Both work under fully observable regimes. Yin and Zhou (2003) work with an investor who minimizes the variance of a given fixed expected terminal wealth and does not consume. The other paper with explicit solution is Sotomayor and Cadenillas (2009): in a consumption-portfolio problem which maximizes the expected total discounted utility of consumption, they find optimal exact portfolio and consumption policies to the log-investor and to the power investor. All market parameters (interest rate, stock drift and volatility) are constant within a regime. They work under infinite horizon and hence are not able to capture and analyze horizon effects[2].

In this paper, we propose an approximate analytical solution for the general finite horizon problem with consumption, in which we disentangle risk aversion from the elasticity of intertemporal substitution using stochastic differential utility, and allow for predictability ($\mathfrak{p}_t$)in

[1] Regimes have been used in asset pricing in continuous time in Buffington and Elliott (2002).

[2] Under unobservable regimes, Honda (2003) finds a closed-form analytical solution to the consumption and portfolio problem only for the case where the investor has a constant relative risk aversion equal to 0.5. For all other values, he uses the martingale approach to numerically solve for the optimal policies with Monte Carlo simulation.

a multi-regime economy[3]. The regimes are hidden, and investors have to estimate the probability $(\pi_t)$of each state at each point in time. Our general solution admits a wealth-separable solution of the form:

$$V(W_t, p_t, \pi_t, \tau) = H(p_t, \pi_t, \tau)\frac{W_t^{1-\gamma}}{1-\gamma}, \tag{1}$$

in which $\tau = T - t$ is the time left till the final horizon. We find an approximate linear expression for $H(p_t, \pi_t, \tau)$ in terms of the state variables p_t, π_t and their squares. The coefficients multiplying the state variables are horizon-dependent and solve a system of ordinary differential equations.

The optimal portfolio strategy contains the usual myopic allocation and two hedging demands related to the predictor and to the regime probabilities, all inversely proportional to the coefficient of risk aversion. The three components depend also implicitly on the elasticity of intertemporal substitution (EIS). Optimal consumption is affected by the state variables through the function $H(p_t, \pi_t, \tau)$ and is directly impacted by the EIS.

We show in a simple two-regime setting that the approximate formulas are accurate with respect ot the optimal strategy obtained by simulation. We then use the parameters of a model similar to Guidolin and Timmermann (2007), with four regimes in stock and bond returns, to explore several features of our approximate solution. We analyze dynamic strategies when regimes are assumed to be known or when the investor needs to estimate the probabilities of being in each of the states. We consider both equal and ergodic (long-run) probabilities. When the regime is known, the horizon does not seem to play a role in the portfolios since the strategies, given the current state of the economy, are almost constant. This implies that hedging demands due to regime changes are very small. Indeed we verify that the total wealth lost when ignoring hedging demands due to regime switches is negligible.

When stock returns are predictable by the dividend yield, the dynamic strategies remain stable when the regimes are known. However, hedging demands with respect to the predictor are now important, especially for stocks because of a positive correlation with dividend yields. We also show that hedging demands increase with high values for the predictor.

When adding intermediate consumption in the problem, we conclude that the consumption-to-wealth ratio does not vary much with the predictor and with the regimes.

[3] Campani and Garcia (2017) have developed a similar approximation method in an intertemporal setting without regimes. The main focus was to study the optimal portfolio and consumption strategies for an investor with recursive utility and a finite horizon.

Several other papers are related to our work. Graflund and Nilsson (2003) study the relevance of intertemporal hedging and regimes under a dynamic portfolio problem with no consumption in which the investor maximizes power utility from terminal wealth. The discrete-time setting includes one risky asset and a riskless bond, in which the investor rebalances the portfolio monthly. The problem is again solved numerically, conditioned on the current (and observable) regime, with the number of regimes being determined with a Monte Carlo likelihood ratio test. The results are that ignoring the regimes for long-horizon investors is costly, while intertemporal hedging is present in some regimes but not in others.

The importance of regime shifts while modeling asset returns has been examined by a number of researchers. For example, Ang and Bekaert (2004) show the importance of regime switching models in tactical asset allocation. Tu (2010), in a similar study, concludes that certainty-equivalent losses associated with ignoring regime switching in portfolio decisions are generally above 2% per year. Guidolin and Hyde (2010) show that vector autoregressive models cannot capture regime shifts in asset returns. They use a three-regime model in which the investor has power utility. All these papers are set in discrete-time and solve the problem numerically, via Monte Carlo simulation methods.

Guidolin and Timmermann (2006, 2008) use a discrete-time regime switching model for the asset returns to solve the portfolio problem (with no consumption), in which the investor has utility over moments of terminal wealth distribution. The current regime is observable and the approximate optimal portfolio weights are found as the roots of a system of polynomial equations (first-order conditions). Under unobservable regimes, Guidolin and Timmermann (2005, 2007) specify a four-state regime switching model in discrete-time with a richer risky asset menu (one large-stock index, one small-stock index and a ten-year bond). Their optimal choices, under power time-additive utility, are found numerically, with Monte Carlo techniques.

The rest of the paper is structured as follows. Section 2 describes the economy and the investor's problem. The solution is explained in Section 3 and its accuracy is assessed in a simple setting in Section 4. Section 5 sets up a four-regime model for small and large stocks, long-term government bonds and Treasuries, computing optimal portfolios for various scenarios. Section 6 introduces a predictor and shows how it affects optimal strategies in the same four-regime economy. The case with intermediate consumption is discussed in Section 7 while Section 8 concludes. An appendix provides all the necessary details about the analytical solution as well as the estimation and simulation procedures to apply our methodology and assess its accuracy.

2 Setting

We set up a continuous-time model in a frictionless and arbitrage-free financial market which embeds $n+1$ different available assets: the short-term riskless asset (say, a Treasury bill) and n different risky assets.

We will denote the short-term riskless asset price at time t as M_t and it will follow the deterministic process below:

$$\frac{dM_t}{M_t} = rdt, \tag{2}$$

in which the instantaneous interest rate r is assumed to be constant[4].

The economy is ruled by a regime-switching unobservable state variable Y_t. This state variable is an independent continuous-time Markov chain, right-continuous and admitting only values in $R = \{1, 2, ..., m\}$, where R represents the finite set of m possible regimes in the economy. The regime-switching process Y_t, starting at any random time t_0 on a given state remains there for an exponentially distributed length of time, and then jumps to another state[5]. More precisely, given that the current state is i, the probability of jumping to another state j over the next Δt time period is $P_{ij,\Delta t} = \frac{\lambda_{ij}}{\sum\limits_{k \neq i} \lambda_{ik}} \left(1 - e^{-\sum\limits_{k \neq i} \lambda_{ik} \Delta t}\right)$, with $j \neq i \in R$ and $\lambda_{ij} \geq 0$. We define $\lambda_{ii} = -\sum\limits_{k \neq i} \lambda_{ik} \leq 0$ such that $P_{ij,\Delta t} = \frac{\lambda_{ij}}{-\lambda_{ii}} \left(1 - e^{\lambda_{ii} \Delta t}\right)$. Straightforwardly, the probability of *staying* in the same regime i over the next Δt time period is $P_{ii,\Delta t} = e^{\lambda_{ii} \Delta t}$. The parameters λ_{ij} ($j \neq i \in R$) are assumed constant and represent the density of transition probabilities from regime i to regime j: to see this point, just notice that $Lim_{\Delta t \to 0} \frac{P_{ij,\Delta t}}{\Delta t} = \lambda_{ij}$. The closer to zero λ_{ii} is, the more persistent is regime i. When $\lambda_{ii} = 0$, once the economy jumps to state i, it will remain there forever. The present model can then be understood as a generalization of the standard single regime model, since this particular case is nested if we assume $\lambda_{ii} = 0$ and that the initial state of the economy is already state i. Another standard intrinsic assumption of our model is that the inter-regime times are independent and also independent of all risky assets Brownian motions.

The vector π_t is defined as an m x 1 column vector storing the probabilities of being in each

[4] This is a common assumption in portfolio models with regime switchings. See, for example, Guidolin and Timmermann (2007), Liu (2011) or Honda (2003).

[5] A very important property of the exponential distribution is that it is memoryless, in accordance to the Markov property. This *memorylessness* can be stated as follows: $P(T > s + t | T > s) = P(T > t)$ for all $s, t \geq 0$.

possible economy state at time t, conditioned on the available information at the same time t:

$$\pi_t = \begin{bmatrix} \pi_{1,t} & \pi_{2,t} & \dots & \pi_{m,t} \end{bmatrix}^T. \tag{3}$$

In this market, the investor is *small*, in the sense that her attitudes do not affect market prices, being thus only a price taker. Because the states are unobservable, we assume that the investor filters the n risky assets dynamics as weighted average processes, in which the probabilities just defined are the weights:

$$\frac{\mathbf{dS_t}}{\mathbf{S_t}} = \mathbf{D_{s,t}}\pi_t dt + (\mathbf{V}\pi_t)\, \mathbf{dZ_t}. \tag{4}$$

In equation 4, we need to define its elements. We begin by defining $\frac{\mathbf{dS_t}}{\mathbf{S_t}}$ as a column vector with all n risky assets processes $(\frac{dS_{j,t}}{S_{j,t}})$. The matrices $\sigma_{\mathbf{s,i}}$, which are constant and specific to the regime $(i \in R)$, are defined as below[6]:

$$\sigma_{\mathbf{s,i}} = \begin{bmatrix} \sigma_{11,i} & 0 & \dots & 0 \\ \sigma_{21,i} & \sigma_{22,i} & \dots & 0 \\ \dots & \dots & \dots & 0 \\ \sigma_{n1,i} & \sigma_{n2,i} & \dots & \sigma_{nn,i} \end{bmatrix}, \tag{5}$$

Now we define $\mathbf{V}$ as a 1 x m row vector of matrices such as below[7]:

$$\mathbf{V} = \begin{bmatrix} \sigma_{\mathbf{s,1}} & \sigma_{\mathbf{s,2}} & \dots & \sigma_{\mathbf{s,m}} \end{bmatrix}. \tag{6}$$

Still in equation 4, $\mathbf{D_{s,t}}$ represents a (time-varying) matrix of dimension n x m that depends on a given predictor p_t:

$$\mathbf{D_{s,t}} = \underbrace{\begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,m} \\ a_{2,1} & a_{2,2} & \dots & a_{2,m} \\ \dots & \dots & \dots & \dots \\ a_{n,1} & a_{n,2} & \dots & a_{n,m} \end{bmatrix}}_{\mathbb{A}} + \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_n \end{bmatrix}}_{\mathbb{B}} \underbrace{\begin{bmatrix} 1 & 1 & \dots & 1 \end{bmatrix}}_{m \text{ columns}} p_t, \tag{7}$$

[6] These regime-dependent volatility matrices are designed to be lower triangular with absolutely no loss of generality.
[7] Alternatively, we have also tested a model in which the investor averages the variance-covariance matrices, instead of the volatility matrices. However, the results are virtually the same.

in which the coefficients $a_{j,i}$, b_j are constant for every $i \in R$ and $j = 1, 2, ..., n$. On its turn, $\mathbf{dZ_t}$ is an $n \times 1$ column vector whose elements are standard and independent Brownian motion processes given by $Z_{1,t}$, $Z_{2,t}$, ..., and $Z_{n,t}$. The predictor process is assumed to be filtered by the investor as follows:

$$dp_t = \mathbf{D_p}\pi_t dt + \underbrace{\begin{bmatrix} \sigma_{p1} & \sigma_{p2} & \dots & \sigma_{pn} & \sigma_{pp} \end{bmatrix}}_{\sigma_p} * \underbrace{\begin{bmatrix} dZ_{1,t} & dZ_{2,t} & \dots & dZ_{n,t} & dZ_{p,t} \end{bmatrix}^T}_{\mathbf{dZ_t^*}}, \tag{8}$$

in which $\sigma_\mathbf{p}$ is a constant row vector with $n+1$ elements, $dZ_{p,t}$ is a standard Brownian motion which is independent of all previously defined Brownian motions. On its turn, the drift above is a constant row vector defined as:

$$\mathbf{D_p} = \begin{bmatrix} \mu_{p,1} & \mu_{p,2} & \dots & \mu_{p,m} \end{bmatrix}. \tag{9}$$

It is important to notice that the standard single-state model is obtained from our model if $\pi_\mathbf{t} = [1 \quad 0 \quad \dots \quad 0]^T$. In such case, equations 4 and 8 reflect standard processes usually used by the available literature in single-state models. In particular, the risky assets will follow the standard geometric Brownian motion, in which the drifts depend linearly on the predictor. Regarding the process followed by the predictor, the investor averages out the drifts of each state (using each current state probability), and on the other hand she observes a constant volatility across the states: this assumption was crucial to keep the problem tractable.

The state probabilities, given by $\pi_\mathbf{t}$, will be treated as state variables in our model. It is completely determined from the risky assets returns and from the predictor, such that we assume it follows the process below:

$$\underbrace{\begin{bmatrix} d\pi_{1,t} \\ d\pi_{2,t} \\ \dots \\ d\pi_{m,t} \end{bmatrix}}_{d\pi_t} = \underbrace{\begin{bmatrix} \sum_{i=1}^{m} \lambda_{i1}\pi_{i,t} \\ \sum_{i=1}^{m} \lambda_{i2}\pi_{i,t} \\ \dots \\ \sum_{i=1}^{m} \lambda_{im}\pi_{i,t} \end{bmatrix}}_{\mu_{\pi,t}} dt + \underbrace{\begin{bmatrix} \sigma_{11,\pi} & \sigma_{12,\pi} & \dots & \sigma_{1n,\pi} & \sigma_{1p,\pi} \\ \sigma_{21,\pi} & \sigma_{22,\pi} & \dots & \sigma_{2n,\pi} & \sigma_{2p,\pi} \\ \dots & \dots & \dots & \dots & \\ \sigma_{m1,\pi} & \sigma_{m2,\pi} & \dots & \sigma_{mn,\pi} & \sigma_{mp,\pi} \end{bmatrix}}_{\sigma_\pi} \mathbf{dZ_t^*}, \tag{10}$$

with the suitable definition $\sigma_{\mathbf{i},\pi} = \begin{bmatrix} \sigma_{i1,\pi} & \sigma_{i2,\pi} & \dots & \sigma_{in,\pi} & \sigma_{ip,\pi} \end{bmatrix}$ as a row vector for all $i \in R$[8].

[8] In fact, one of the probability processes is redundant, given that all probabilities must sum up to one.

The drift above was chosen in accordance with theorem 9.1 of Liptser and Shiryaev (2000).

The investor chooses consumption C_t optimally at every time t. The variable C_t is defined as the rate of consumption at the instant t, such that the investor consumes $C_t dt$ over the time interval from t to $t + dt$. She invests a fraction $\alpha_{1,t}$ of her total wealth W_t in the risky asset 1, and so on till a fraction $\alpha_{n,t}$ in the risky asset n, while the rest is put in the short-term riskless bond. Her wealth dynamics will thus write:

$$\begin{aligned} dW_t &= (1 - \alpha_t \mathbf{1}) W_t \frac{dM_t}{M_t} + W_t \alpha_t \frac{\mathbf{dS_t}}{\mathbf{S_t}} - C_t dt \\ &= (1 - \alpha_t \mathbf{1}) W_t r dt + W_t \alpha_t [\mathbf{D_{s,t}} \pi_t dt + (\mathbf{V} \pi_t) \mathbf{dZ_t}] - C_t dt \\ &= W_t r dt + W_t \alpha_t [(\mathbf{D_{s,t}} \pi_t - r\mathbf{1}) dt + (\mathbf{V} \pi_t) \mathbf{dZ_t}] - C_t dt, \end{aligned} \tag{11}$$

where $\mathbf{1}$ represents a column vector of n ones and $\alpha_t = [\alpha_{1,t} \ \alpha_{2,t} \ \dots \ \alpha_{n,t}]$.

Our investor's preferences are over consumption and terminal wealth (bequest), represented by a continuous-time recursive utility function:

$$J_t = E_t \left[\int_{u=t}^{T} f(C_u, J_u) du + \frac{W_T^{1-\gamma}}{1-\gamma} \right]. \tag{12}$$

Duffie and Epstein (1992) proved that this formulation, also known by stochastic differential utility, is valid with bequest. Time T is the horizon of our investor: Because our goal is to be closer to reality as possible, we are mostly interested on the life-cycle problem, although the infinite horizon problem can be studied as a nested case of our framework (obtained as the limit when the horizon $T \to \infty$). Our investor seeks for optimal investment and consumption policies from now through time T in the future. What happens after time T does not affect at all this investor.

In equation (12), $f(C, J)$ is a normalized aggregator of current rate of consumption and continuation utility that takes the following form:

$$f(C, J) = \frac{\beta}{1 - \frac{1}{\psi}} (1 - \gamma) J \left\{ \left[\frac{C}{[(1-\gamma) J]^{\frac{1}{1-\gamma}}} \right]^{1 - \frac{1}{\psi}} - 1 \right\}. \tag{13a}$$

In this equation, $\beta > 0$ is a rate of time preference. As it is well known by now, γ controls investor's attitudes over states of nature (*i.e.*, states of the economy) while ψ is related to investor's consumption choices over time. We will usually refer to them as (relative) risk aversion

However, we prefer to keep this redundancy for sake of clarity.

and elasticity of intertemporal substitution parameters respectively.

There are two special cases of the normalized aggregator given by equation (13a): $\psi = \frac{1}{\gamma}$ and $\psi = 1$. The first case is merely the standard, time additive power utility function[9] (from which log-utility optimal choices will be obtained as the limit of the respective optimal choices when $\gamma \to 1$). In the second case, interpreted as the limit when $\psi \to 1$, the aggregator takes the following limit form:

$$\psi = 1 \qquad \to \qquad f(C, J) = \beta (1-\gamma) J \ln \left\{ \frac{C}{[(1-\gamma) J]^{\frac{1}{1-\gamma}}} \right\}. \tag{13b}$$

The case in which $\psi = 1$ is special because it refers to the particular situation in which the investor consumes myopically, in the sense that investment opportunities will not impact this policy. When $\psi < 1$, income effects will prevail and better opportunities will increase consumption. When $\psi > 1$, substitution effects will dominate because the investor is more willing to postpone consumption and hence better take advantages of improved investment opportunities.

3 Solving the Problem

The Bellman equation to the recursive framework problem will be:

$$0 = \sup_{\{C_t, \alpha_t\}} \left[\begin{array}{c} f(C_t, J_t) + \frac{\partial J_t}{\partial t} + J_w \left[W_t r + W_t \boldsymbol{\alpha_t} \left(\mathbf{D_{s,t}} \boldsymbol{\pi_t} - r\mathbf{1} \right) - C_t \right] + \\ + \frac{1}{2} J_{ww} W_t^2 \boldsymbol{\alpha_t} \left(\mathbf{V} \boldsymbol{\pi_t} \right) \left(\mathbf{V} \boldsymbol{\pi_t} \right)^{\mathbf{T}} \boldsymbol{\alpha}_{\mathbf{t}}^{\mathbf{T}} + J_p \mathbf{D_p} \boldsymbol{\pi_t} + \frac{1}{2} J_{pp} \boldsymbol{\sigma_p} \boldsymbol{\sigma}_{\mathbf{p}}^{\mathbf{T}} + \\ + \sum_{i=1}^{m} J_{\pi_i} \sum_{j=1}^{m} \lambda_{ji} \pi_{j,t} + \frac{1}{2} \sum_{i=1}^{m} J_{\pi_i \pi_i} \boldsymbol{\sigma_{i,\pi}} \boldsymbol{\sigma}_{\mathbf{i},\pi}^{\mathbf{T}} + J_{wp} W_t \boldsymbol{\alpha_t} \left(\mathbf{V} \boldsymbol{\pi_t} \right) \overline{\boldsymbol{\sigma}}_{\mathbf{p}}^{\mathbf{T}} + \\ + \sum_{i=1}^{m} J_{w\pi_i} W_t \boldsymbol{\alpha_t} \left(\mathbf{V} \boldsymbol{\pi_t} \right) \overline{\boldsymbol{\sigma}}_{\mathbf{i},\pi}^{\mathbf{T}} + \sum_{i=1}^{m} J_{p\pi_i} \boldsymbol{\sigma_p} \boldsymbol{\sigma}_{\mathbf{i},\pi}^{\mathbf{T}} + \sum_{i<j} J_{\pi_i \pi_j} \boldsymbol{\sigma_{i,\pi}} \boldsymbol{\sigma}_{\mathbf{j},\pi}^{\mathbf{T}} \end{array} \right], \tag{14}$$

where $\overline{\boldsymbol{\sigma}}_{\mathbf{p}}^{\mathbf{T}}$ and $\overline{\boldsymbol{\sigma}}_{\mathbf{i},\pi}^{\mathbf{T}}$ represent the transposed vector of respectively $\boldsymbol{\sigma_p}$ and $\boldsymbol{\sigma_{i,\pi}}$ $(i \in R)$ without their last element[10]. Moreover, $f(C, J)$ is given by equation (13a) if $\psi \neq 1$ or (13b) if $\psi = 1$. The subscripts denote partial derivatives (except t, which merely denotes the value at time t). The function $V(W_t, p_t, \boldsymbol{\pi_t}, t) = \sup_{\{C_t, \boldsymbol{\alpha_t}\}} [J(W_t, p_t, \boldsymbol{\pi_t}, t)]$ is the value function for this problem and it depends on the observable state variables of the economy.

[9] Note that the resulting formulation may not take the standard power utility form, but will imply the same underlying preferences, and hence the same consumption and asset allocation choices.

[10] This is just an adjustment to make matrix dimensions agree. After all, these last terms refer to an independent Brownian motion with no correlation with the wealth dynamics.

The first-order condition for consumption in the recursive problem is given by:

$$V_w = \frac{\partial f(C_t, V_t)}{\partial C_t} = \beta(1-\gamma)^{\frac{\frac{1}{\psi}-\gamma}{1-\gamma}} V_t^{\frac{\frac{1}{\psi}-\gamma}{1-\gamma}} C_t^{-\frac{1}{\psi}}, \tag{15a}$$

from which we solve for the optimal consumption strategy and find that:

$$C_t = \beta^{\psi} V_w^{-\psi} \left[(1-\gamma) V_t\right]^{\frac{1-\psi\gamma}{1-\gamma}}, \tag{15b}$$

which reduces to $C_t = \beta V_w^{-1} V_t (1-\gamma)$ when $\psi = 1$. We can plug equation (15b) into (13a) and (13b) to write the recursive aggregator as function only of the value function itself:

$$f(V_t) = \frac{1}{1-\frac{1}{\psi}} \left\{\beta^{\psi} V_w^{1-\psi} \left[(1-\gamma) V_t\right]^{\frac{1-\psi\gamma}{1-\gamma}} - \beta(1-\gamma) V_t\right\} \qquad \psi \neq 1, \tag{16a}$$

$$f(V_t) = \beta(1-\gamma) V_t \ln\left\{\beta V_w^{-1} \left[V_t (1-\gamma)\right]^{\frac{-\gamma}{1-\gamma}}\right\} \qquad \psi = 1. \tag{16b}$$

The first-order condition for the portfolio weight gives the following:

$$\begin{aligned} \alpha_{\mathbf{t}} &= \frac{V_w}{-V_{ww} W_t} \left(\mathbf{D}_{\mathbf{s},\mathbf{t}} \pi_{\mathbf{t}} - r\mathbf{1}\right)^{\mathbf{T}} \left[(\mathbf{V}\pi_{\mathbf{t}})(\mathbf{V}\pi_{\mathbf{t}})^{\mathbf{T}}\right]^{-\mathbf{1}} + \\ &+ \frac{V_{wp}}{-V_{ww} W_t} \overline{\sigma}_{\mathbf{p}} (\mathbf{V}\pi_{\mathbf{t}})^{-\mathbf{1}} + \sum_{i=1}^{m} \frac{V_{w\pi_i}}{-V_{ww} W_t} \overline{\sigma}_{\mathbf{i},\pi} (\mathbf{V}\pi_{\mathbf{t}})^{-\mathbf{1}}. \end{aligned} \tag{17}$$

We can easily identify the myopic and hedging demand terms in the equation above. The second and third portfolio components (the hedging demands) will be held by the investor to hedge against future undesirable movements respectively in the predictor and in the state probabilities. We now substitute the optimal expressions for consumption and portfolio strategies into equation (14) to obtain the final Bellman equation for the problem under recursive utility:

$$\begin{aligned} 0 &= f(V_t) + \frac{\partial V_t}{\partial t} + V_w W_t r - \beta^{\psi} V_w^{1-\psi} \left[(1-\gamma) V_t\right]^{\frac{1-\gamma\psi}{1-\gamma}} + V_p \mathbf{D}_{\mathbf{p}} \pi_{\mathbf{t}} + \frac{\mathbf{1}}{\mathbf{2}} V_{pp} \sigma_{\mathbf{p}} \sigma_{\mathbf{p}}^{\mathbf{T}} + \\ &+ \sum_{i=1}^{m} V_{p\pi_i} \sigma_{\mathbf{p}} \sigma_{\mathbf{i},\pi}^{\mathbf{T}} + \sum_{i,j=1}^{m} V_{\pi_i} \lambda_{ji} \pi_{j,t} + \frac{1}{2} \sum_{i,j=1}^{m} \left(V_{\pi_i \pi_j} \sigma_{\mathbf{i},\pi} \sigma_{\mathbf{j},\pi}^{\mathbf{T}} - \frac{V_{w\pi_i} V_{w\pi_j}}{V_{ww}} \overline{\sigma}_{\mathbf{i},\pi} \overline{\sigma}_{\mathbf{j},\pi}^{\mathbf{T}}\right) - \\ &- \frac{1}{2} \frac{V_w^2}{V_{ww}} \left(\mathbf{D}_{\mathbf{s},\mathbf{t}} \pi_{\mathbf{t}} - r\mathbf{1}\right)^{\mathbf{T}} \left[(\mathbf{V}\pi_{\mathbf{t}})(\mathbf{V}\pi_{\mathbf{t}})^{\mathbf{T}}\right]^{-1} \left(\mathbf{D}_{\mathbf{s},\mathbf{t}} \pi_{\mathbf{t}} - r\mathbf{1}\right) - \frac{1}{2} \frac{V_{wp}^2}{V_{ww}} \overline{\sigma}_{\mathbf{p}} \overline{\sigma}_{\mathbf{p}}^{\mathbf{T}} - \\ &- \frac{V_w V_{wp}}{V_{ww}} \overline{\sigma}_{\mathbf{p}} (\mathbf{V}\pi_{\mathbf{t}})^{-\mathbf{1}} \left(\mathbf{D}_{\mathbf{s},\mathbf{t}} \pi_{\mathbf{t}} - r\mathbf{1}\right) - \sum_{i=1}^{m} \frac{V_w V_{w\pi_i}}{V_{ww}} \overline{\sigma}_{\mathbf{i},\pi} (\mathbf{V}\pi_{\mathbf{t}})^{-\mathbf{1}} \left(\mathbf{D}_{\mathbf{s},\mathbf{t}} \pi_{\mathbf{t}} - r\mathbf{1}\right) - \\ &- \sum_{i=1}^{m} \frac{V_{wp} V_{w\pi_i}}{V_{ww}} \overline{\sigma}_{\mathbf{p}} \overline{\sigma}_{\mathbf{i},\pi}^{\mathbf{T}}. \end{aligned} \tag{18}$$

We also consider a problem with no consumption in which all (power) utility comes from terminal wealth. The Bellman equation in this case is (slightly) simplified and details can be found at appendix A. Both problems admit a wealth-separable solution of the form:

$$V(W_t, p_t, \pi_t, \tau) = H(p_t, \pi_t, \tau)\, \frac{W_t^{1-\gamma}}{1-\gamma}, \tag{19}$$

in which $\tau = T - t$ is the time left to horizon and terminal condition states that $H(p_t, \pi_t, 0) = 1$. The bad news is that it is virtually impossible to find the exact functional form of $H(p_t, \pi_t, \tau)$. However, with some approximation techniques (see appendix B), we can provide an approximation for H as follows:

$$H(p_t, \pi_t, \tau) = \exp\left[\begin{array}{c} A_0(\tau) + A_p(\tau)\, p_t + \sum_{i=1}^{m} A_i(\tau)\, \pi_{i,t} + B_p(\tau)\, p_t^2 + \\ + \sum_{i=1}^{m} B_i(\tau)\, \pi_{i,t}^2 + \sum_{i=1}^{m} C_{pi}(\tau)\, p_t \pi_{i,t} + \sum_{i<j} C_{ij}(\tau)\, \pi_{i,t}\pi_{j,t} \end{array}\right]. \tag{20}$$

All coefficients above are time-varying and solve a system of ordinary differential equations with boundary conditions equal to zero when $\tau = 0$ (*i.e.*, at maturity). This solution is valid for both the cases with and without consumption, even though the coefficients will be different in each situation. The optimal portfolio strategy will always be given by:

$$\begin{aligned} \alpha_t \;=\; & \frac{1}{\gamma}\left(\mathbf{D}_{s,t}\pi_t - r\mathbf{1}\right)^{\mathbf{T}} \left[(\mathbf{V}\pi_t)(\mathbf{V}\pi_t)^{\mathbf{T}}\right]^{-1} + \\ & + \frac{1}{\gamma}\left[A_p(\tau) + 2B_p(\tau)\, p_t + \sum_{i=1}^{m} C_{pi}(\tau)\, \pi_{i,t}\right] \overline{\sigma}_{\mathbf{p}} (\mathbf{V}\pi_t)^{-1} + \\ & + \frac{1}{\gamma}\sum_{i=1}^{m}\left[A_i(\tau) + 2B_i(\tau)\, \pi_{i,t} + C_{pi}(\tau)\, p_t + \sum_{j\neq i} C_{ij}(\tau)\, \pi_{j,t}\right] \overline{\sigma}_{\mathbf{i},\pi} (\mathbf{V}\pi_t)^{-1}. \end{aligned} \tag{21}$$

The first term above refers to the myopic allocation while both others are the hedging demands related to the predictor and to the probabilities, respectively. Observe that all terms are proportional to the reciprocal of γ, supporting the interpretation of this parameter as the relative risk aversion coefficient also in the recursive framework. Our model implies that the Sharpe ratio is calculated using the risk premia and the volatility matrices as weighted averages of their respective values in the regimes, with the filtered probabilities as weights.

In the appendix, one can note that when $\gamma = 1$, the solution to the system implies that all coefficients are constant and equal to zero, except $A_0 = -\beta\tau$. As a consequence, the optimal portfolio is myopic (no hedging demands), as expected. Another nested case is the single state model: in such case, π_t is constant and therefore $\sigma_{i,\pi}$ is zero (no hedging demands related to regime changes), leaving the investor with the same portfolio that a single state investor would see as optimal.

When we have consumption, the optimal strategy is obtained substituting the value function into the respective first-order condition, given by (15b):

$$\frac{C_t}{W_t} = \beta^{\psi} H^{\frac{1-\psi}{1-\gamma}}. \tag{22}$$

We observe that consumption is affected by the state variables (through function H) and by ψ as well. By the way, the portfolio allocation also depends on this parameter through the coefficients because the system to determine them has all equations depending on ψ. These optimal strategies nest the ones when $\psi = 1$: in such a case, the consumption-to-wealth ratio remains constant and equal to β. In other words, the investor with unitary elasticity of intertemporal substitution will consume just as the investor who sees a single regime in the economy, *i.e.*, with a constant proportional rate, even when the predictor evolves or the regime seems to be switching to a better (or worse) one in terms of opportunities: this is due to the fact that $\psi = 1$ implies the cancelation of the income and substitution effects in a way such that the opportunity set will not impact investor's consumption. Current literature says that the investor consumes myopically[11].

4 Evaluating the accuracy of the approximate solution

In this section, we adapt our model by discretizing the decision process. We adopt a discrete-time strategy in which the investor makes her portfolio and consumption decisions based on current information and waits until the next period to remake decisions with the new acquired information. The smaller is this period, the closer is the model to real-time rebalancing decisions, but of course investors have to account for transaction costs. We set the rebalancing period equal to one month to follow the portfolio literature on discrete-time regime switching models (*e.g.*, Guidolin and Timmermann (2005, 2007, 2008), Ang and Timmermann (2012), Ang and Bekaert (2002, 2004)).

In appendix C, we show the details to estimate the parameters needed to implement our

[11] One could argue that the problem with no consumption is nested in the problem with consumption when $\psi \to \infty$. In such case, consumption will vanish and the portfolio strategy will converge to the one obtained when consumption is not allowed.

model, using Hamilton (1989) methodology. What is still left for estimation is the matrix σ_π, introduced by the model used in this study. We explain now the procedure to estimate this matrix. We firstly create an $(n+1) \times m$ matrix (denoted by $\mathbf{D_t}$) consisting of $\mathbf{D_{s,t}}$ in the first n rows and of $\mathbf{D_p}$ over the last row. Then we calculate the monthly time-series for the drifts $(\mathbf{D_t}\pi_\mathbf{t})$ and volatility matrix $(\mathbf{V}\pi_\mathbf{t})$ as seen by the investor[12]. We are therefore able to estimate the discretized process below, which follows from equations (4) and (8):

$$\Delta\mathbf{Z}^*_\mathbf{t} = \sigma_\mathbf{t}{}^{-\mathbf{1}} \left(\mathbf{LR_t} - \mathbf{D_t}\pi_\mathbf{t}\right), \tag{23}$$

where $\mathbf{LR_t}$ is an $(n+1) \times 1$ vector of the n risky assets observed log-returns at period t together with the predictor change in the same period (see appendix C for more details). The matrix $\sigma_\mathbf{t}$ is the concatenation of $(\mathbf{V}\pi_\mathbf{t})$ with $\sigma_\mathbf{p}$, thus creating a square matrix of order $n+1$ (the n first elements of the last column are filled up with zeros). We use equation (10) to write the discretization $(\Delta\pi_\mathbf{t} - \mu_{\pi,\mathbf{t}}) = \sigma_\pi \Delta\mathbf{Z}^*_\mathbf{t}$ and store its left-hand time-series in an $m \times T$ matrix denoted by $(\Delta\pi - \mu_\pi)$, as well as we store all increments $\Delta\mathbf{Z}^*_\mathbf{t}$ in an $(n+1) \times T$ matrix denoted simply by $\Delta\mathbf{Z}^*$ (where T is the time-series length). We finally obtain the desired estimation as follows:

$$\sigma_\pi = (\Delta\pi - \mu_\pi)\, \Delta\mathbf{Z}^{*\mathbf{T}} \left(\Delta\mathbf{Z}^* \Delta\mathbf{Z}^{*\mathbf{T}}\right)^{-\mathbf{1}}. \tag{24}$$

To evaluate whether our dynamic model delivers good approximations of the optimal choices, we carry out a Monte Carlo simulation-based method to find optimal choices. We then compare these portfolios to the ones resulting from our approximate solution. Our measure of accuracy will be the wealth equivalent utility loss due to our (sub-optimal) solution when compared to the (optimal) simulated solution. This loss is calculated as follows: we consider two identical investors, except that they invest different initial amounts and that the first follows the optimal strategy while the second follows the approximate strategy. The second investor begins with a wealth equal to \$100. Given their strategies, we match their value functions (*i.e.*, utilities), thus being able to calculate the actual initial wealth the first investor needs to have. This wealth will be less than \$100, since this investor follows an optimal strategy. The initial wealth difference will be the percentage wealth equivalent utility loss due to the sub-optimal strategy. If this loss is negligible, it means that the model provides accurate strategies and the investor can hence

[12] To achieve a more robust methodology, we disregarded the first year of data. This is explained by the fact that the choice of the starting probabilities still had some effects over the filtered probabilities during this first year. These starting probabilities were revealed to be completely irrelevant after a year from the starting date.

rely on it.

For the sake of simplicity, we analyze here the problem with no consumption and no predictability[13]. In other words, the problem with no consumption keeps the simulation process simple while capturing all assumptions and approximations we need to assess:

I The way the investor interprets the risky assets, predictor and probabilities dynamics (equations (4), (8) and (10)), given that she does not observe the regimes;

II The discretization of our continuous-time model into Hamilton discrete-time regime switching model (equation (33) or (34a) and (34b)) ; and

III The volatility approximation (equation (29)) in the Bellman equation of the problem.

The simulation[14] burden quickly increases with almost all variables. Therefore, to keep the simulation process feasible, we choose $N = 30,000$ simulations[15] and a setting with two risky assets (along with the riskless asset) and 2 regimes[16]. The horizon was set from 1 month to 10 years.

Our investor is considering two risky assets, represented by a stock index and a long-term bond index. The stock index is represented by the S&P500 index, dividends included. The long-term bond index is represented by the CRSP portfolio of 10-year T-bonds[17].

The parameter values are shown in Tables 1 and 2[18]. The interest rate is the average T-Bill yield of 5.3% per year and investor's relative risk aversion coefficient is set equal to 5.

We report the optimal dynamic strategies in Table 3. We also report the monthly wealth equivalent loss due to our approximate solution when compared to the optimal strategy, obtained through simulation. For horizons from one month to ten years, we observed that all losses are less than 0.01% for the cases in which the next period regime is either known or unknown (in this case, we show the results using ergodic and equal probabilities). In fact, to confirm the

[13] Including consumption has a limited impact over the optimal portfolio. Furthermore, the approximation needed when $\psi \neq 1$ was discussed in Campani and Garcia (2017) and shown to be accurate. Including predictability does not add any further approximation and there is no reason to believe that predictability will affect the model accuracy. Moreover, with predictability, the simulation burden increases considerably.

[14] Appendix D explains details of the simulation process.

[15] Guidolin and Timmermann (2007) say that $N \geq 20,000$ guarantees sufficient accuracy.

[16] The number of assets and the number of regimes affect directly the simulation burden since the numbers of elements in the portfolio and probability grids increase exponentially and slow down considerably the simulation process. Furthermore, if we were to consider predictability, we would need a third grid (to the predictor), overloading even more the simulation process

[17] Both data series were downloaded from CRSP through WRDS. The data is monthly and from January, 1952 (after the 1951 Treasury Accord) till December, 2011, encompassing 720 observations .

[18] We also report in the first table the (myopic) portfolio obtained with the single regime model for an investor with $\gamma = 5$. This portfolio is constant for all horizons once the investor believes that the parameters are constant and that a single regime rules the economy.

robustness of such accuracy, we analyzed the losses for every point in a grid with a 10% step for the probabilities and the maximum loss was less than 0.016%.

5 An Illustration: A Four-Regime Model with Large Firms, Small Firms and Long-Term Government Bonds

In this section, we still maintain the absence of return predictability and intermediate consumption.

5.1 Model Specification

Given the accuracy of our approximate solutions, we can now easily analyze the implications of a more elaborate portfolio problem with more assets and regimes. In this section, to illustrate our analytical model, we follow Guidolin and Timmermann (2007) and apply it with $n = 3$ risky assets (a large cap index, a small cap index and a long-term bond index) and $m = 4$ regimes[19].

The parameter values are shown in Table 4[20]. As before, the interest rate is the average T-Bill yield of 5.3% per year and investor's rate of time preference is set at a yearly rate of $\beta = 2\%$. Regime 1 is characterized by high volatilities and negative expected returns for large and small caps. Regime 2 is characterized by low positive returns with relatively low volatilities. In regime 3, stocks perform better (especially small caps) with higher volatilities and the long-term bond performs slightly worse than in the previous regime, but with a very low standard deviation. Observe that regimes 2 and 3 are important to identify size effects in stock returns since they clearly differentiate large and small caps performances. Finally, regime 4 is a very optimistic state, in which all risky assets present very high expected returns. Moreover, stocks feature lower volatilities when compared to regime 1, as opposed to the bond asset. In fact, the large caps present a much higher expected return when compared to the bond asset in this regime, with only a slightly higher volatility. The small caps have the highest expected return in this state, but also the highest volatility. We also note interesting correlation changes across

[19] These authors show that a model with fewer states is clearly misspecified. We use the same terminology for the four regimes: crash, slow growth, bull and recovery states although the magnitude of the mean returns change in our longer sample (January 1952 to December 2011). The large and small cap indices are the highest and lowest quintiles based on the capitalization value of all NYSE, AMEX, and NASDAQ firms. This data was downloaded from Kenneth French's website. The long-term bond index is represented by the CRSP portfolio of 10-year T-bonds, downloaded from CRSP through WRDS. The data include 720 monthly observations.

[20] We also report in this table the (myopic) portfolio obtained with the single regime model for an investor with $\gamma = 5$. This portfolio is constant for all horizons once the investor believes that the parameters are constant and that a single regime rules the economy.

regimes: for example, the correlation between both stock classes can be as low as 0.12 in regime 4 or as high as 0.83 in regime 1. The correlation between bonds and small caps can be negative as in regimes 2 and 4 or positive as in regimes 1 and 3.

We report in Figure 1 the filtered probabilities for the estimation period. They will be used to compute the dynamic investment strategies since they reflect the information available to investors when they make their portfolio and consumption decisions.

The transition probabilities show that regime 1 is almost surely followed by regime 4, lasting on average only 3 months. Regime 2, the most persistent one lasting 16 months on average, is almost surely followed by regime 1. Regime 3 lasts on average for 9 months and it is more likely to be followed by regimes 1 or 2. Regime 4, the most optimistic one, is (unfortunately) the least persistent (average duration of only 2 months) and it can be followed by any of the other regimes (although regime 1 is the most likely one). The ergodic probabilities show that over the long term (steady state), regime 1 appears in 21.5% of the periods, while regime 2 is the most common one, showing up in half of the periods. Regimes 3 and 4 are less frequent, presenting respectively 11.8% and 16.5% of chances. In Table 5 we report in the last two columns the ergodic probabilities and average durations for all the regimes[21].

5.2 Dynamic Strategies

Figure 2 shows the dynamic strategies for different horizons and a known current state as given by the model presented in this article. As expected, the current state is key for determining the optimal weights in the available assets. If the current regime is 1, the investor sells short the large stocks to invest the rest in bonds (40%) and in the riskless rate (260%): this is the crash regime, so the investor sells short risky assets to buy safer assets. Large stocks are preferred to be sold instead of small stocks because these have a much higher volatility. In regime 2, large stocks are the main component of the portfolio and the investor sells short small caps and bonds, since bonds have negative risk premium in this regime and small caps present a lower expected return with a higher volatility when compared to large stocks. In regime 3, long-term bonds are hugely sold short to mainly invest in the riskless rate (1550%) and also in stocks (280% in total): in this regime, as in the previous one, bonds have a lower expected return than the risk-free

[21] Because one of the probability processes is redundant, we must have that all processes in (10) add up to zero. By definition of the λ's, we see that this is clear for the drift. However, we should also have that the columns of σ_π add up to zero. This can be proven from (24): the values obtained by our estimation procedure, shown in Table 5, underline this point.

rate. Regime 4 is a bullish state where the investor borrows to load up on stocks.

Figure 3 plots the dynamic strategies if the current regime is unknown. In this case, the investor needs to estimate the probabilities of being in each of the states: We consider in these plots equal and ergodic (long-run) probabilities. We observe that uncertainty significantly affects the way investors form their portfolios. When information about the current regime is not clear, the strategies are not so aggressive: uncertainty does not allow the investor to track the market such as before. Another interesting result is that small stocks enter the portfolio with a greater (positive) weight than large stocks in both cases: this is a result of the fact that small caps are preferred in 3 of the 4 regimes (large stocks are the first choice only in regime 2). This fact highlights the importance of splitting the stock allocation between large and small stocks. It is also important to point out the use of long-term bonds, which represent an important fraction of the portfolio under uncertainty on the current regime.

Are Hedging Demands Important? One aspect of all strategies in both figures discussed in this subsection is that horizon seems to play almost no role in the portfolios: the strategies, given the current state of the economy, are almost constant. Therefore, hedging demands due to regime changes appear very small. As a consequence, one important question is whether hedging demands can be ignored (or not). In other words, we want to analyze whether hedging demands are really important to dynamic investors in the multi-regime economy without predictability. We answer this issue by assessing the wealth equivalent loss one investor incurs when ignoring hedging demands. To achieve this goal, we start back on the Bellman equation (26) and imagine a myopic investor who follows the myopic portfolio strategy (*i.e.*, without hedging demands) as given by:

$$\alpha_{\mathbf{t}} = \frac{1}{\gamma}\left(\mathbf{D}_{\mathbf{s},\mathbf{t}}\pi_{\mathbf{t}} - \mathbf{r}\mathbf{1}\right)^{\mathbf{T}}\left[\left(\mathbf{V}\pi_{\mathbf{t}}\right)\left(\mathbf{V}\pi_{\mathbf{t}}\right)^{\mathbf{T}}\right]^{-\mathbf{1}} \tag{25}$$

We follow exactly the same steps as in the optimal problem and the solution (which is sub-optimal) is given by the same value function as before, but with different coefficients[22]. As this value function is sub-optimal, it will provide lower values than those provided by the optimal value function, making possible the comparison. To find the total wealth equivalent loss, we equate the optimal value function (in which initial wealth is unknown) to the myopic sub-optimal one (in which wealth is standardized to 100) to solve for the unknown initial wealth

[22] These coefficients solve a similar (although simpler) system of equations, which we show in the online appendix. The non-predictability case there is nested when $\mathbb{B}$ is a zero column matrix.

of the optimal strategy. The difference to 100 gives the total percentage wealth equivalent loss due to the myopic strategy.

We computed the total wealth equivalent loss for the myopic strategy starting from each of the four regimes, from ergodic and from equal probabilities for horizons from 1 month to 10 years and the maximum loss was less than 0.005% per month[23]. We can safely conclude that the costs of ignoring hedging demands due to switches in regimes are very low, if not negligible.

6 Including Predictability

In this section, we include predictability as in the general model detailed in Section 2. Following the literature (*e.g.*, Guidolin and Timmermann (2007), Campbell and Viceira (1999)and Campbell et al. (2003) among others), we choose the dividend yield as predictor of asset returns (dividend yield over the Standard & Poors 500 Composite Index). We take the natural logarithm of the percentage values in order to make the predictor consistent with the process given by (8)[24]. The time period analyzed, all other time series and constants (β and the riskless rate for example) remain the same as in the case without predictability. Including the predictor affects the parameter estimates, as reported in Tables 6 and 7.

To facilitate the comparison with the case without predictability, we include in the first table the mean returns of each asset under each regime. These mean returns are calculated using the mean predictor value under each of the regimes[25].

Indeed, we can observe that in general predictability changes the magnitude of the parameters but keeps the overall nature of the regimes. Under regime 1, stocks have very bad expected returns (especially small caps), and long-term bonds remain the good choice for the investor. This regime is still the most volatile one. However, predictability worsened the stock performance and improved the bond one. Indeed, with predictability, regime 1 is less persistent (two-month average duration) and less frequent (ergodic probability of 10.4%). It is more of a crash state than without predictability. At the other extreme, regime 4 remains the most bullish state, in which stocks have the highest expected returns with volatilities lower than in regime 1 (but higher than in regimes 2 and 3). However, with predictability, the performance in this regime is

[23] Even if we expand the horizon up to 40 years, this value stays the same.
[24] This means that the predictor data series should admit negative values given that its process is not exponential.
[25] To calculate the mean predictor values in each regime, we opted to use the smoothed probabilities instead of the filtered (real-time) ones in order to achieve the best possible estimation.

somewhat worse than before, with lower expected returns for all risky assets.

Regimes 2 and 3 remain the least volatile regimes. In regime 2 large stocks outperform small stocks with a higher expected return and a lower volatility just as with no predictability. In regime 3, small stocks have higher expected returns when compared to large stocks, but higher volatility, exactly as in the case without predictability. Nevertheless, just as with regimes 1 and 4, the stock performances worsen (especially for small caps), but the regimes are more persistent (both have 21 months of average duration). The joint ergodic probability increases from 62.0% to 67.2%. Bonds, however, have better expected returns, but higher volatilities. Moreover, under regime 4, their expected returns are now lower than they were without predictability. Regimes 2 and 3 are also differentiated by the correlation between bonds and stocks: while in the slow growth state, bonds have fairly positive correlations with small and large caps (0.28 and 0.46 respectively), the bull state exhibits negative correlations between bonds and stocks (-0.18 and -0.17 respectively).

Regarding transition probabilities, regime 1 continues to be followed almost surely by the recovery state, which by its turn can be followed by any of the other regimes (although roughly half of the chances are for regime 1, just as before). Regime 2 is again almost surely followed by regime 1, but now regime 3 is more likely (89%) to be followed by regime 1 (instead of by regime 2 in the case with no predictability).

In Figure 4, the filtered probabilities confirm that regimes 3 and 4 are more persistent and more common than without predictability. The opposite can be said about regime 1. Regime 2 is singular: while it is more frequent in the problem without predictability, slow growth markets are on average more persistent with predictability. It is also interesting to note that the information provided by the predictor allows investors to make more accurate inferences about the regimes in the economy.

6.1 Dynamic Strategies with Predictability

To study the new features that predictability adds to the dynamic portfolio allocation problem, we start analyzing how the shares evolve as function of investor's horizon when the predictor is set at its overall sample average ($\overline{p}$). Figures 5 and 6 present the plots respectively when the current regime is know and unknown.

If the current regime is supposed to be known, the risky assets strategies are stable for horizons up to 10 years, at least when $p_t = \overline{p}$ (however, notice that the riskless asset presents slight downwards curves when horizon increases). Overall, predictability changed the way investors see the four regimes (with different parameters) and, as a consequence, changed the way investors form their portfolios. For example, under regime 1, investors will buy a sizeable amount of large stocks, which sounds counter-intuitive since this is the crash state. But notice that the investor will sell-short an even larger amount of small stocks: in fact, she is taking advantage of the high correlation between these two stock groups and the fact that small caps have a much worse performance in this state. Doing so, investors are trying to benefit from the singularities of the crash state. Given the very bad performance of small stocks, investors sell them to buy mostly bonds but also large stocks. Why also large stocks and not just bonds, which have their best expected return in this state? Just because bonds have a low correlation (0.19) with small stocks, while large stocks present a very high one (0.79).

Predictability has also changed the way investors choose their portfolio if they are uncertain about the current regime. With equal current probabilities, large stocks have a positive share in the optimal portfolio but, as before, this is due to the negative share of small stocks. Since the probability of regime 1 is high, long-term bonds are the preferred assets. If the probabilities are the steady-state ones, all three risky assets are prominent in the portfolio.

6.2 Sensitivity Analysis with respect to the Predictor

To understand how the portfolios change with different values for the predictor (state variable), we analyzed the shares for a broad range of predictor values. To illustrate the results, we plot in Figures 7 to 10 how the portfolios evolve for the same current states as in all previous plots when the predictor is two standard deviations above and below its historical mean.

We see that small stocks are the most sensitive risky asset[26]. Observe for example that under state 3, the shares invested in small caps can vary from -60% if the predictor is well below its mean to over 200%, once the predictor is well above its mean.

Overall, stocks will represent a larger share of the portfolio when the value of the predictor increases, irrespective of the state probabilities, since stock expected returns have a positive correlation with the dividend yield. From the plots, we can also observe that hedging demands are stronger with high predictor values.

[26] This result was also found by Guidolin and Timmermann (2007).

6.3 Predictability and Hedging Demands

Using the same methodology as before, we calculate the expected wealth equivalent loss a dynamic myopic investor suffers in the beginning of the investment period for ignoring hedging demands. If on one hand, the magnitude of the losses has clearly increased with predictability, on the other hand, the importance of hedging demands will depend on the predictor's value, as suggested by the previous subsection. We analyzed it on the whole range of predictor values and could observe that hedging demands are important when the predictor value increases. For instance, if the predictor is equal to its sample average, the total loss is always less than 0.73%, which can be considered very low for a 10-year strategy. However, with the predictor two standard deviations above, this loss can reach 4.35% for the same strategy. Table 8 presents these results. With more extreme predictor values such as three standard deviations above its mean, the total loss to a ten-year strategy reaches about 6% for all regimes and for equal or ergodic probabilities. If the horizon increases to 40 years, two standard deviations above the mean will result in losses of about 30% of initial wealth, regardless of the initial regime. As the predictor value increases, expected returns for stocks go up and the share of stocks in the portfolio increases through hedging demands.

7 Including Consumption

The main effects comes from the investor's elasticity of intertemporal substitution (ψ). Campani and Garcia(2017) discussed in detail the role of ψ in a setting with predictability. Therefore, in this section, we want to determine how the presence of multiple regimes affects the consumption pattern of investors. We report in Figure 11 how optimal consumption evolves with and without predictability, for realistic values for ψ and the predictor[27]. The left plots refer to the problem without predictability, while the right ones are for the problem with predictability. In the upper plots, we show the consumption policy as a function of the horizon for each regime and when the investor uses ergodic or equal probabilities (the predictor is set at its historical mean, ψ is fixed at 0.75 and γ remains, as always in this paper, equal to 5). We can observe that consumption is roughly equal in both problems with and without predictability. Moreover, and somehow surprisingly, consumption policy barely changes with the regime.

The lower plots present the real-time exercise of how a 12-month horizon investor would con-

[27] We also analyzed the impact of including consumption on the portfolio strategy and concluded that it was small with and without predictability. Therefore, all previous results hold also with intermediate consumption.

sume with the actual filtered probabilities and predictor values from January 1980 to December 2011. In these graphs, we show five different investors: two willing to postpone consumption ($\psi = 1.25$ and $\psi = 1.5$), two others that hardly substitute consumption today for tomorrow ($\psi = 0.5$ and $\psi = 0.75$) and the myopic (in terms of consumption) investor ($\psi = 1$). These two plots are important because they use a broad real-time range for current probabilities and for the predictor. They confirm that consumption-to-wealth ratio variations with the predictor and with the regimes are very limited.

8 Conclusion

This paper derives an approximate analytical solution to a portfolio problem under stochastic differential utility (recursive utility in continuous time) in a multi-regime economy. We show in a two-state economy that the approximation is very accurate. To understand the implications of multiple regimes, we study a four-state economy with four classes of assets (large and small caps, long-term government bonds and a riskless asset), following Guidolin and Timmermann (2007). We confirm that the current regime or its filtered probabilities, given that the states are unobservable, have a considerable impact on the portfolio allocation. All four assets enter the portfolio at various degrees depending on the regimes. The large and small stocks play a distinct role in the portfolio.

Calculating wealth equivalent losses, we show that hedging demands (horizon effects) were completely negligible in the economy with no predictability. However, including predictability of the expected returns increases hedging demands, making them important for high values of the predictor. Predictability plays an important role in the make-up of the optimal portfolio. Therefore, tracking the predictor is crucial, since optimal portfolios will be strongly impacted by the current predictor value. Allowing investors to consume does not change the outlook in any way. The consumption-to-wealth ratio policy, unlike the portfolio policy, depends only marginally on the filtered probabilities of the current regime and on the predictor. It is mainly determined by parameters such as the elasticity of intertemporal substitution and the investor's horizon.

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Appendices

A The Bellman Equation for the Problem With No Consumption

The Bellman equation to the recursive framework problem with no consumption will be:

$$0 = \sup_{\{\alpha_t\}} \left[\begin{array}{c} -\beta J_t + \frac{\partial J_t}{\partial t} + J_w W_t \left[r + \alpha_\mathbf{t} \left(\mathbf{D_{s,t}} \pi_\mathbf{t} - r\mathbf{1} \right) \right] + \\ + \frac{1}{2} J_{ww} W_t^2 \alpha_\mathbf{t} \left(\mathbf{V} \pi_\mathbf{t} \right) \left(\mathbf{V} \pi_\mathbf{t} \right)^\mathbf{T} \alpha_\mathbf{t}^\mathbf{T} + J_p \mathbf{D_p} \pi_\mathbf{t} + \frac{\mathbf{1}}{\mathbf{2}} J_{pp} \sigma_\mathbf{p} \sigma_\mathbf{p}^\mathbf{T} + \\ + \sum_{i=1}^m J_{\pi_i} \sum_{j=1}^m \lambda_{ji} \pi_{j,t} + \frac{1}{2} \sum_{i=1}^m J_{\pi_i \pi_i} \sigma_{\mathbf{i},\pi} \sigma_{\mathbf{i},\pi}^\mathbf{T} + J_{wp} W_t \alpha_\mathbf{t} \left(\mathbf{V} \pi_\mathbf{t} \right) \overline{\sigma}_\mathbf{p}^\mathbf{T} + \\ + \sum_{i=1}^m J_{w\pi_i} W_t \alpha_\mathbf{t} \left(\mathbf{V} \pi_\mathbf{t} \right) \overline{\sigma}_{\mathbf{i},\pi}^\mathbf{T} + \sum_{i=1}^m J_{p\pi_i} \sigma_\mathbf{p} \sigma_{\mathbf{i},\pi}^\mathbf{T} + \sum_{i<j} J_{\pi_i \pi_j} \sigma_{\mathbf{i},\pi} \sigma_{\mathbf{j},\pi}^\mathbf{T} \end{array} \right]. \tag{26}$$

Substituting the optimal expression for the portfolio weight, as given by 17, the final Bellman equation turns out to be as follows:

$$\begin{aligned} 0 \;=\; & -\beta V_t + \frac{\partial V_t}{\partial t} + V_w W_t r + V_p \mathbf{D_p} \pi_\mathbf{t} + \frac{\mathbf{1}}{\mathbf{2}} V_{pp} \sigma_\mathbf{p} \sigma_\mathbf{p}^\mathbf{T} + \\ & + \sum_{i=1}^m V_{p\pi_i} \sigma_\mathbf{p} \sigma_{\mathbf{i},\pi}^\mathbf{T} + \sum_{i,j=1}^m V_{\pi_i} \lambda_{ji} \pi_{j,t} + \frac{1}{2} \sum_{i,j=1}^m \left(V_{\pi_i \pi_j} \sigma_{\mathbf{i},\pi} \sigma_{\mathbf{j},\pi}^\mathbf{T} - \frac{V_{w\pi_i} V_{w\pi_j}}{V_{ww}} \overline{\sigma}_{\mathbf{i},\pi} \overline{\sigma}_{\mathbf{j},\pi}^\mathbf{T} \right) - \\ & - \frac{1}{2} \frac{V_w^2}{V_{ww}} \left(\mathbf{D_{s,t}} \pi_\mathbf{t} - r\mathbf{1} \right)^\mathbf{T} \left[\left(\mathbf{V} \pi_\mathbf{t} \right) \left(\mathbf{V} \pi_\mathbf{t} \right)^\mathbf{T} \right]^{-\mathbf{1}} \left(\mathbf{D_{s,t}} \pi_\mathbf{t} - r\mathbf{1} \right) - \frac{1}{2} \frac{V_{wp}^2}{V_{ww}} \overline{\sigma}_\mathbf{p} \overline{\sigma}_\mathbf{p}^\mathbf{T} - \\ & - \frac{V_w V_{wp}}{V_{ww}} \overline{\sigma}_\mathbf{p} \left(\mathbf{V} \pi_\mathbf{t} \right)^{-\mathbf{1}} \left(\mathbf{D_{s,t}} \pi_\mathbf{t} - r\mathbf{1} \right) - \sum_{i=1}^m \frac{V_w V_{w\pi_i}}{V_{ww}} \overline{\sigma}_{\mathbf{i},\pi} \left(\mathbf{V} \pi_\mathbf{t} \right)^{-\mathbf{1}} \left(\mathbf{D_{s,t}} \pi_\mathbf{t} - r\mathbf{1} \right) - \\ & - \sum_{i=1}^m \frac{V_{wp} V_{w\pi_i}}{V_{ww}} \overline{\sigma}_\mathbf{p} \overline{\sigma}_{\mathbf{i},\pi}^\mathbf{T}. \end{aligned} \tag{27}$$

B Details to Find the Approximate Analytical Solution

The partial differential equation to be solved comes from substituting the value function given by (19) into the final Bellman equation (18) (or (27)). We also guess that $H\left(p_t, \pi_\mathbf{t}, \tau\right) = e^{g(p_t, \pi_\mathbf{t}, \tau)}$

to get the following equation:

$$
\begin{aligned}
0 \;=\;& \hat{f}(g) - \frac{\partial g}{\partial \tau} + (1-\gamma)\, r + \frac{\partial g}{\partial p}\mathbf{D_p}\pi_\mathbf{t} + \frac{1}{2}\left[\frac{\partial^2 g}{\partial p^2} + \left(\frac{\partial g}{\partial p}\right)^2\right]\sigma_\mathbf{p}\sigma_\mathbf{p}^\mathbf{T} + \\
&+ \sum_{i=1}^{m}\left(\frac{\partial^2 g}{\partial p \partial \pi_i} + \frac{\partial g}{\partial p}\frac{\partial g}{\partial \pi_i}\right)\sigma_\mathbf{p}\sigma_{\mathbf{i},\pi}^\mathbf{T} + \frac{1-\gamma}{\gamma}\sum_{i=1}^{m}\frac{\partial g}{\partial p}\frac{\partial g}{\partial \pi_i}\overline{\sigma}_\mathbf{p}\overline{\sigma}_{\mathbf{i},\pi}^\mathbf{T} + \\
&+ \frac{1}{2}\sum_{i,j=1}^{m}\left[2\lambda_{ji}\pi_{j,t}\frac{\partial g}{\partial \pi_i} + \left(\frac{\partial^2 g}{\partial \pi_i \partial \pi_j} + \frac{\partial g}{\partial \pi_i}\frac{\partial g}{\partial \pi_j}\right)\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{j},\pi}^\mathbf{T}\right] + \\
&+ \frac{1}{2}\frac{1-\gamma}{\gamma}\sum_{i,j=1}^{m}\frac{\partial g}{\partial \pi_i}\frac{\partial g}{\partial \pi_j}\overline{\sigma}_{\mathbf{i},\pi}\overline{\sigma}_{\mathbf{j},\pi}^\mathbf{T} + \\
&+ \frac{1}{2}\frac{1-\gamma}{\gamma}\left(\mathbf{D_{s,t}}\pi_\mathbf{t} - r\mathbf{1}\right)^\mathbf{T}\left[(\mathbf{V}\pi_\mathbf{t})(\mathbf{V}\pi_\mathbf{t})^\mathbf{T}\right]^{-\mathbf{1}}\left(\mathbf{D_{s,t}}\pi_\mathbf{t} - r\mathbf{1}\right) + \\
&+ \frac{1}{2}\frac{1-\gamma}{\gamma}\left(\frac{\partial g}{\partial p}\right)^2\overline{\sigma}_\mathbf{p}\overline{\sigma}_\mathbf{p}^\mathbf{T} + \frac{1-\gamma}{\gamma}\frac{\partial g}{\partial p}\overline{\sigma}_\mathbf{p}(\mathbf{V}\pi_\mathbf{t})^{-\mathbf{1}}\left(\mathbf{D_{s,t}}\pi_\mathbf{t} - r\mathbf{1}\right) + \\
&+ \frac{1-\gamma}{\gamma}\sum_{i=1}^{m}\frac{\partial g}{\partial \pi_i}\overline{\sigma}_{\mathbf{i},\pi}(\mathbf{V}\pi_\mathbf{t})^{-\mathbf{1}}\left(\mathbf{D_{s,t}}\pi_\mathbf{t} - r\mathbf{1}\right),
\end{aligned}
\tag{28}
$$

where:

$$
\begin{aligned}
\hat{f}(g) &= -\beta && \textit{with no consumption,} \\
\hat{f}(g) &= \frac{1-\gamma}{\psi-1}\left(\beta^\psi e^{\frac{1-\psi}{1-\gamma}g} - \beta\psi\right) && \textit{when } \psi \neq 1, \textit{ and} \\
\hat{f}(g) &= (1-\gamma)\,\beta\left[\ln\left(\beta e^{\frac{-1}{1-\gamma}g}\right) - 1\right] && \textit{when } \psi = 1.
\end{aligned}
$$

The equation above is very complicated to handle mainly because of the time-varying term $(\mathbf{V}\pi_\mathbf{t})$, which is the kernel of the asset return volatilities as seen by the investor. Therefore, we will consider this term constant and given by its long-run (ergodic) value[28]:

$$
(\mathbf{V}\pi_\mathbf{t}) \approx (\mathbf{V}\pi_\infty)\,. \tag{29}
$$

[28] The so called ergodic probabilities are the unconditional expectation of the probabilities.

The PDE for function g becomes:

$$\begin{aligned}
0 \;=\;& \hat{f}(g) - \frac{\partial g}{\partial \tau} + (1-\gamma)\, r + \frac{\partial g}{\partial p}\mathbf{D_p}\pi_\mathbf{t} + \frac{1}{2}\left[\frac{\partial^2 g}{\partial p^2} + \left(\frac{\partial g}{\partial p}\right)^2\right]\sigma_\mathbf{p}\sigma_\mathbf{p}^\mathbf{T} + \\
&+ \sum_{i=1}^{m}\left(\frac{\partial^2 g}{\partial p \partial \pi_i} + \frac{\partial g}{\partial p}\frac{\partial g}{\partial \pi_i}\right)\sigma_\mathbf{p}\sigma_{\mathbf{i},\pi}^\mathbf{T} + \frac{1-\gamma}{\gamma}\sum_{i=1}^{m}\frac{\partial g}{\partial p}\frac{\partial g}{\partial \pi_i}\overline{\sigma}_\mathbf{p}\overline{\sigma}_{\mathbf{i},\pi}^\mathbf{T} + \\
&+ \frac{1}{2}\sum_{i,j=1}^{m}\left[2\lambda_{ji}\pi_{j,t}\frac{\partial g}{\partial \pi_i} + \left(\frac{\partial^2 g}{\partial \pi_i \partial \pi_j} + \frac{\partial g}{\partial \pi_i}\frac{\partial g}{\partial \pi_j}\right)\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{j},\pi}^\mathbf{T}\right] + \\
&+ \frac{1}{2}\frac{1-\gamma}{\gamma}\sum_{i,j=1}^{m}\frac{\partial g}{\partial \pi_i}\frac{\partial g}{\partial \pi_j}\overline{\sigma}_{\mathbf{i},\pi}\overline{\sigma}_{\mathbf{j},\pi}^\mathbf{T} + \\
&+ \frac{1}{2}\frac{1-\gamma}{\gamma}\left(\mathbf{D_{s,t}}\pi_\mathbf{t} - r\mathbf{1}\right)^\mathbf{T}\left[\left(\mathbf{V}\pi_\infty\right)\left(\mathbf{V}\pi_\infty\right)^\mathbf{T}\right]^{-\mathbf{1}}\left(\mathbf{D_{s,t}}\pi_\mathbf{t} - r\mathbf{1}\right) + \\
&+ \frac{1}{2}\frac{1-\gamma}{\gamma}\left(\frac{\partial g}{\partial p}\right)^2\overline{\sigma}_\mathbf{p}\overline{\sigma}_\mathbf{p}^\mathbf{T} + \frac{1-\gamma}{\gamma}\frac{\partial g}{\partial p}\overline{\sigma}_\mathbf{p}\left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}}\left(\mathbf{D_{s,t}}\pi_\mathbf{t} - r\mathbf{1}\right) + \\
&+ \frac{1-\gamma}{\gamma}\sum_{i=1}^{m}\frac{\partial g}{\partial \pi_i}\overline{\sigma}_{\mathbf{i},\pi}\left(\mathbf{V}\pi_\infty\right)^{-\mathbf{1}}\left(\mathbf{D_{s,t}}\pi_\mathbf{t} - r\mathbf{1}\right),
\end{aligned} \tag{30}$$

which allows us to guess the following analytical solution:

$$\begin{aligned}
g\left(p_t, \pi_\mathbf{t}, \tau\right) \;=\;& A_0(\tau) + A_p(\tau)\, p_t + \sum_{i=1}^{m} A_i(\tau)\,\pi_{i,t} + B_p(\tau)\, p_t^2 + \\
&+ \sum_{i=1}^{m} B_i(\tau)\,\pi_{i,t}^2 + \sum_{i=1}^{m} C_{pi}(\tau)\, p_t \pi_{i,t} + \sum_{i<j} C_{ij}(\tau)\,\pi_{i,t}\pi_{j,t},
\end{aligned} \tag{31}$$

in which all coefficients are time-varying and depend on the primitive parameters of the model describing the regime-switching economy, market opportunities and investor's preferences. Substitution into the equation above and also using equation (7) will render the following (where

$C_{ji,j>i} = C_{ij}$ by definition):

$$
\begin{aligned}
0 \;=\; & f(g) - A_0' - A_p' p_t - \sum_{i=1}^{m} A_i' \pi_{i,t} - B_p' p_t^2 - \sum_{i=1}^{m} B_i' \pi_{i,t}^2 - \sum_{i=1}^{m} C_{pi}' p_t \pi_{i,t} - \\
& - \sum_{i<j} C_{ij}' \pi_{i,t} \pi_{j,t} + (1-\gamma) r + \left(A_p + 2B_p p_t + \sum_{i=1}^{m} C_{pi} \pi_{i,t} \right) \mathbf{D_p} \boldsymbol{\pi_t} + \\
& + \frac{1}{2} \left[2B_p + \left(A_p + 2B_p p_t + \sum_{i=1}^{m} C_{pi} \pi_{i,t} \right)^2 \right] \sigma_{\mathbf{p}} \sigma_{\mathbf{p}}^{\mathbf{T}} + \\
& + \sum_{i=1}^{m} \left[C_{pi} + \left(A_p + 2B_p p_t + \sum_{k=1}^{m} C_{pk} \pi_{k,t} \right) \left(A_i + 2B_i \pi_{i,t} + C_{pi} p_t + \sum_{k \neq i} C_{ik} \pi_{k,t} \right) \right] \sigma_{\mathbf{p}} \sigma_{\mathbf{i},\pi}^{\mathbf{T}} + \\
& + \frac{1-\gamma}{\gamma} \sum_{i=1}^{m} \left(A_p + 2B_p p_t + \sum_{k=1}^{m} C_{pk} \pi_{k,t} \right) \left(A_i + 2B_i \pi_{i,t} + C_{pi} p_t + \sum_{k \neq i} C_{ik} \pi_{k,t} \right) \overline{\sigma}_{\mathbf{p}} \overline{\sigma}_{\mathbf{i},\pi}^{\mathbf{T}} + \\
& + \sum_{i,j=1}^{m} \lambda_{ji} \pi_{j,t} \left(A_i + 2B_i \pi_{i,t} + C_{pi} p_t + \sum_{k \neq i} C_{ik} \pi_{k,t} \right) + \sum_{i=1}^{m} B_i \sigma_{\mathbf{i},\pi} \sigma_{\mathbf{i},\pi}^{\mathbf{T}} + \sum_{i<j} C_{ij} \sigma_{\mathbf{i},\pi} \sigma_{\mathbf{j},\pi}^{\mathbf{T}} + \\
& + \frac{1}{2} \sum_{i,j=1}^{m} \left(A_i + 2B_i \pi_{i,t} + C_{pi} p_t + \sum_{k \neq i} C_{ik} \pi_{k,t} \right) \left(A_j + 2B_j \pi_{j,t} + C_{pj} p_t + \sum_{k \neq j} C_{jk} \pi_{k,t} \right) \sigma_{\mathbf{i},\pi} \sigma_{\mathbf{j},\pi}^{\mathbf{T}} + \\
& + \frac{1}{2} \frac{1-\gamma}{\gamma} \sum_{i,j=1}^{m} \left(A_i + 2B_i \pi_{i,t} + C_{pi} p_t + \sum_{k \neq i} C_{ik} \pi_{k,t} \right) \left(A_j + 2B_j \pi_{j,t} + C_{pj} p_t + \sum_{k \neq j} C_{jk} \pi_{k,t} \right) \overline{\sigma}_{\mathbf{i},\pi} \overline{\sigma}_{\mathbf{j},\pi}^{\mathbf{T}} + \\
& + \frac{1}{2} \frac{1-\gamma}{\gamma} \left(\mathbb{A} \boldsymbol{\pi_t} + \mathbb{B} p_t - r\mathbf{1} \right)^{\mathbf{T}} \left[(\mathbf{V} \pi_\infty) (\mathbf{V} \pi_\infty)^{\mathbf{T}} \right]^{-\mathbf{1}} \left(\mathbb{A} \boldsymbol{\pi_t} + \mathbb{B} p_t - r\mathbf{1} \right) + \\
& + \frac{1}{2} \frac{1-\gamma}{\gamma} \left(A_p + 2B_p p_t + \sum_{i=1}^{m} C_{pi} \pi_{i,t} \right)^2 \overline{\sigma}_{\mathbf{p}} \overline{\sigma}_{\mathbf{p}}^{\mathbf{T}} + \\
& + \frac{1-\gamma}{\gamma} \left(A_p + 2B_p p_t + \sum_{i=1}^{m} C_{pi} \pi_{i,t} \right) \overline{\sigma}_{\mathbf{p}} (\mathbf{V} \pi_\infty)^{-\mathbf{1}} \left(\mathbb{A} \boldsymbol{\pi_t} + \mathbb{B} p_t - r\mathbf{1} \right) + \\
& + \frac{1-\gamma}{\gamma} \sum_{i=1}^{m} \left(A_i + 2B_i \pi_{i,t} + C_{pi} p_t + \sum_{k \neq i} C_{ik} \pi_{k,t} \right) \overline{\sigma}_{\mathbf{i},\pi} (\mathbf{V} \pi_\infty)^{-\mathbf{1}} \left(\mathbb{A} \boldsymbol{\pi_t} + \mathbb{B} p_t - r\mathbf{1} \right).
\end{aligned}
\tag{32}
$$

For the problems with no consumption or with $\psi = 1$, this equation can be solved by matching equivalent terms, such that we find a system of $\frac{m^2}{2} + \frac{5m}{2} + 3$ equations with exactly the same number of (time-varying) unknowns (the coefficients of function g). The boundary condition for all coefficients is zero when $\tau = 0$. In the online appendix to this paper, we present the system of equations which is solved with standard software. For the general problem in which $\psi \neq 1$, this equation is non-linear, what prevents us to solve it analytically. However, we use the same log-linearization approximation technique explained in Campani and Garcia (2017) and are then able to solve it. The details of this general solution can also be found in the online appendix.

With the value function known, the optimal consumption and portfolio policies are thus just a consequence of the first-order conditions given by (15b) and (17).

C Estimation of Parameters

We assume that a monthly period, used in our paper, is short enough such that we can reasonably match Hamilton's transition probabilities with our densities of transition probabilities as follows[29]:

$$\text{Prob}\{Y_{t+1} = j | Y_t = i\} = P_{ij,\Delta t=1} = P_{ij} = \frac{\lambda_{ij}}{-\lambda_{ii}}\left(1 - e^{\lambda_{ii}}\right), \tag{33}$$

in which we have conveniently chosen the time unit as the same as the period length (that is, one month). We will then have the following identities:

$$\lambda_{ii} = \ln P_{ii}, \; and \tag{34a}$$

$$\lambda_{ij} = -\frac{P_{ij}\ln P_{ii}}{1 - P_{ii}}. \tag{34b}$$

We collect the (constant) discrete-time probabilities in a matrix we call $\mathbf{P}$:

$$\mathbf{P} = \begin{bmatrix} P_{11} & P_{12} & \dots & P_{1m} \\ P_{21} & P_{22} & \dots & P_{2m} \\ \dots & \dots & \dots & \dots \\ P_{m1} & P_{m2} & \dots & P_{mm} \end{bmatrix}. \tag{35}$$

The investor uses the filter explained in Hamilton (1989) to infer the current regime and take her optimal consumption and investment decisions. This optimal inference relies on iterating the pair of equations below:

$$\widehat{\mathbf{Y}}_{\mathbf{t|t}} = \frac{\widehat{\mathbf{Y}}_{\mathbf{t|t-1}} \circ \eta_{\mathbf{t}}}{\mathbf{1^T}\left(\widehat{\mathbf{Y}}_{\mathbf{t|t-1}} \circ \eta_{\mathbf{t}}\right)}, \; and \tag{36a}$$

$$\widehat{\mathbf{Y}}_{\mathbf{t+1|t}} = \mathbf{P^T}\widehat{\mathbf{Y}}_{\mathbf{t|t}}, \tag{36b}$$

in which $\mathbf{1}$ represents here an mx1 vector of ones and the symbol $\circ$ denotes the element-by-element multiplication. $\widehat{\mathbf{Y}}_{\mathbf{t|t}}$ and $\widehat{\mathbf{Y}}_{\mathbf{t+1|t}}$ are also m x 1 vectors containing the updated probabilities of having each regime running respectively at times t and $t+1$, given the available information at time t. Finally, $\eta_{\mathbf{t}}$ is another m x 1 vector whose elements are the joint probability densities of the risky assets returns and predictor at time t, conditioned by being on each of

[29] Note that this would be exact if the period considered were the infinitesimally short period dt.

the states. The first equation above uses the current information (*i.e.*, the risky assets returns and predictor value) at time t to update the probabilities of each of the states in this last period t. The second equation then uses this update of the economy regime to optimally estimate the probabilities of being in each state in the next period (next month).

In order to put in practice the iteration above, we need a starting point. Obviously, this can be naturally guessed by the investor, based on her beliefs of the initial state of the economy. In this study, we will start from the long-run regime probabilities (often called ergodic or unconditional probabilities), which are given by:

$$\widehat{\mathbf{Y}}_{\mathbf{1|0}} = \left(A^T A\right)^{-1} A^T e_{m+1}, \tag{37}$$

where e_{m+1} denotes the last column vector of the identity matrix of order $m+1$ and A is an $(m+1)$ x m matrix in which the first m rows are the rows of $I_m - P^T$ (I_m is the identity matrix of order m) and the last row has only 1's[30]. To find vector η_t, we recall that the processes followed by the assets and the predictor, if on a single-regime economy, admit the solutions below[31]:

$$\begin{bmatrix} \ln \frac{S_{1,t+1}}{S_{1,t}} \\ \ln \frac{S_{2,t+1}}{S_{2,t}} \\ \dots \\ \ln \frac{S_{n,t+1}}{S_{n,t}} \\ p_{t+1} - p_t \end{bmatrix} = \begin{bmatrix} \mu_{1,t} - \frac{1}{2}\sum\limits_{k=1}^{n} \sigma_{1k,t}^2 \\ \mu_{2,t} - \frac{1}{2}\sum\limits_{k=1}^{n} \sigma_{2k,t}^2 \\ \dots \\ \mu_{n,t} - \frac{1}{2}\sum\limits_{k=1}^{n} \sigma_{nk,t}^2 \\ \mu_{p,t} \end{bmatrix} + \begin{bmatrix} \sigma_{11,t} & 0 & \dots & 0 & 0 \\ \sigma_{21,t} & \sigma_{22,t} & \dots & 0 & 0 \\ \dots & \dots & \dots & 0 & 0 \\ \sigma_{n1,t} & \sigma_{n2,t} & \dots & \sigma_{nn,t} & 0 \\ \sigma_{p1} & \sigma_{p2} & \dots & \sigma_{pn} & \sigma_{pp} \end{bmatrix} \begin{bmatrix} \Delta Z_{1,t} \\ \Delta Z_{2,t} \\ \dots \\ \Delta Z_{n,t} \\ \Delta Z_{p,t} \end{bmatrix}, \tag{38}$$

such that the element of vector η_t occupying row i is[32]:

$$\eta_{t,i} = \frac{1}{(2\pi)^{\frac{n+1}{2}} \left|\sigma_i \sigma_i^{\mathbf{T}}\right|^{\frac{1}{2}}} \exp\left[-\frac{1}{2}\left(\mathbf{LR_t} - \mathbf{MLR_i}\right)^{\mathbf{T}} \left(\sigma_i \sigma_i^{\mathbf{T}}\right)^{-1} \left(\mathbf{LR_t} - \mathbf{MLR_i}\right)\right], \tag{39}$$

where $\mathbf{LR_t}$ is an $(n+1)$ x 1 vector of the n risky assets observed log-returns at period t together with the predictor change in the same period (*i.e.*, the left-hand side of equation (38)). On its turn, $\mathbf{MLR_i}$ stores the mean of log-returns together with the mean predictor change conditioned

[30] The demonstration of this formula can be found at Hamilton (1989).

[31] We have assumed that any regime switch will take place at the end of period, as it is the standard in discrete-time regime switching models.

[32] Matrix σ_i is an $(n+1)$ x $(n+1)$ matrix built of $\sigma_{s,i}$ concatenated with σ_p in the last row and zeros in the first n rows of its last column, such as in equation (38). The expression $|\sigma_i {\sigma_i}^T|$ denotes the determinant of matrix $\sigma_i \sigma_i^{\mathbf{T}}$.

on the regime:

$$\mathbf{LR_t} = \begin{bmatrix} \ln \frac{S_{1,t+1}}{S_{1,t}} \\ \ln \frac{S_{2,t+1}}{S_{2,t}} \\ \dots \\ \ln \frac{S_{n,t+1}}{S_{n,t}} \\ p_{t+1} - p_t \end{bmatrix} \quad \text{and} \quad \mathbf{MLR_i} = \begin{bmatrix} \mu_{1,i} - \frac{1}{2}\sum_{k=1}^{n} \sigma^2_{1k,i} \\ \mu_{2,i} - \frac{1}{2}\sum_{k=1}^{n} \sigma^2_{2k,i} \\ \dots \\ \mu_{n,i} - \frac{1}{2}\sum_{k=1}^{n} \sigma^2_{nk,i} \\ \mu_{p,i} \end{bmatrix}. \tag{40}$$

The parameter values are estimated using maximum likelihood estimation and the same methodology as in Hamilton (1989).

D Details on the Monte Carlo Simulation Process

We explain in this appendix the details of the Monte Carlo Simulation process designed to assess the accuracy of our approximation technique on Section 4. Let us begin with the dynamic problem, which is by far more complex to solve numerically: Fix the starting point of the problem at $t = 0$, with a dynamic approach (which here means monthly rebalancing of the portfolio) and an horizon set at time T. In this case, the investor's problem at any given moment t is[33]:

$$V\left(W_t, \widehat{\mathbf{Y}}_{\mathbf{t+1|t}}, t\right) = \sup_{\{\alpha_\mathbf{t}, \alpha_{\mathbf{t+1}}, \dots, \alpha_{\mathbf{T-1}}\}} E_t\left[\frac{W_T^{1-\gamma}}{1-\gamma}\right], \qquad t = 0, 1, \dots, T-1. \tag{41}$$

Thinking of date 0, the investor wants to find the optimal asset allocation $\alpha_\mathbf{0}$, given her horizon T, knowing that she will optimally reallocate again her resources monthly from $t = 1$ towards $t = T - 1$. The wealth constraint reads:

$$W_{t+1} = W_t\left[\left(1 - \alpha_\mathbf{t}^\mathbf{T}\mathbf{1}\right)e^r + \alpha_\mathbf{t}^\mathbf{T} e^{\overline{\mathbf{LR}}_\mathbf{t}}\right], \tag{42}$$

in which $e^{\overline{\mathbf{LR}}_\mathbf{t}}$ is a column vector where its elements are the exponentials of the elements of the vector $\mathbf{LR_t}$, defined in Section 4, without its last element (predictor change). Due to power

[33] To be consistent with the notation used in this paper, observe that we use as state variable the filtered probabilities $\widehat{\mathbf{Y}}_{\mathbf{t+1|t}}$ as opposed to $\pi_\mathbf{t}$ because we are under a discrete-time model now. We remind that $\widehat{\mathbf{Y}}_{\mathbf{t+1|t}}$ represents the best guess of the probabilities to the next period, given the information available at the current period. It is calculated using equations (36a) and (36b).

utility (and as shown previously), the value function can be written as:

$$V\left(W_t, \widehat{\mathbf{Y}}_{\mathbf{t+1|t}}, t\right) = \frac{W_t^{1-\gamma}}{1-\gamma} H\left(\widehat{\mathbf{Y}}_{\mathbf{t+1|t}}, t\right), \tag{43}$$

in which terminal condition implies $H\left(\widehat{\mathbf{Y}}_{\mathbf{T+1|T}}, T\right) = 1$. We can couple both equations (41) and (43) to find the recursion below[34]:

$$H\left(\widehat{\mathbf{Y}}_{\mathbf{t+1|t}}, t\right) = \sup_{\{\alpha_t\}} E_t\left[\left(\frac{W_{t+1}}{W_t}\right)^{1-\gamma} H\left(\widehat{\mathbf{Y}}_{\mathbf{t+2|t+1}}, t+1\right)\right]. \tag{44}$$

The recursion above is the key to solve the problem using dynamic programming techniques. We start in the last investor decision, at time $t = T-1$: in this case we know that $H\left(\widehat{\mathbf{Y}}_{\mathbf{T+1|T}}, T\right) = 1$ for all $\widehat{\mathbf{Y}}_{\mathbf{T+1|T}}$ and the right-hand side of the recursion will only depend on wealth's return. The next step is to build a grid of possible values for the probability vector $\widehat{\mathbf{Y}}_{\mathbf{T|T-1}}$: Guidolin and Timmermann (2007) affirmed that five grid discretization points are enough. However, we want to be more conservative and improve the accuracy of the test, so our grid for each element on $\widehat{\mathbf{Y}}_{\mathbf{t+1|t}}$ is chosen to be $\{0, 0.1, 0.2, ..., 1\}$. This larger grid also allows a more detailed robustness check (as we show later on). For each point in this grid[35], we first simulate the next period regime (using the transition probability matrix) and, based on this simulated regime, simulate then the risky assets returns. To be able to calculate the right-hand side of the recursion above, we do this simulation N times and use the law of large numbers to approximate the expectation on it. We proceed choosing the optimal $\alpha_{\mathbf{T-1}}$, which is the one that maximizes the expectation just calculated (and again, this is done for each $\widehat{\mathbf{Y}}_{\mathbf{T|T-1}}$ in the grid: indeed, the optimal strategy at any time t will depend on this state variable). To find this optimal allocation strategy, we again work with a grid for $\alpha_{\mathbf{t}}$, which is chosen to have a 5% step in all assets (a less refined grid leads to inaccurate portfolios). Notice that we have two different grids: one for the probability and another for the portfolio.

So now we have calculated the right-hand side of the recursion above for every $\widehat{\mathbf{Y}}_{\mathbf{T|T-1}}$ in the grid, which means we know $H\left(\widehat{\mathbf{Y}}_{\mathbf{T|T-1}}, T-1\right)$ for every point in the probability grid. We repeat the process backwards until we reach $t = 0$, but at this point we know the value of $\widehat{\mathbf{Y}}_{\mathbf{1|0}}$ such that the optimal portfolio strategy at time 0, given the monthly rebalancings and the horizon T, is determined. One last detail remains to be explained: not at time $t = T-1$, but

[34] An important remark: realistic values for γ, including the one used by this paper, leads to a negative factor $1-\gamma$. In such case, the supreme operator in equation (44) should be read as the minimum operator.
[35] A point of the grid is a full vector $\widehat{\mathbf{Y}}_{\mathbf{t+1|t}}$.

at all proceeding steps, given the grid for $\widehat{\mathbf{Y}}_{\mathbf{t+1|t}}$, nothing guarantees that $\widehat{\mathbf{Y}}_{\mathbf{t+2|t+1}}$ will also be in the grid (this is important because, as pointed out by equation (44), we need the estimated values of $\mathsf{H}\left(\widehat{\mathbf{Y}}_{\mathbf{t+2|t+1}}, \mathsf{t}+1\right)$, but these values were calculated only for points in the grid). We overcome this difficulty choosing $\widetilde{\mathbf{Y}}_{\mathbf{t+2|t+1}}$ as the one in the probability grid which is closest to $\widehat{\mathbf{Y}}_{\mathbf{t+2|t+1}}$ (using the standard Euclidean distance).

E Tables

Table 1: We present in Panel A the parameters and the optimal weights for the single state model. Panel B presents the parameters for the two state model used to assess the accuracy of the approximate solution presented by this paper. The correlation matrices show the volatilities in their diagonals. All data is monthly and $\gamma = 5$.

Panel A: Single State Model	Stocks	Bonds
Mean Returns	0.92%	0.53%
Correlation Matrix - Stocks	0.0426	0.1040
Bonds	0.1040	0.0213
Optimal Weights	52.8%	33.0%
Panel B: Two State Model	Stocks	Bonds
Mean Returns - Regime 1	0.27%	0.98%
Regime 2	1.27%	0.29%
Correlation Matrix - Regime 1		
Stocks	0.0568	0.1173
Bonds	0.1173	0.0298
Correlation Matrix - Regime 2		
Stocks	0.0318	0.1394
Bonds	0.1394	0.0144
Transition Probabilities	Regime 1	Regime 2
Regime 1	88.78%	11.22%
Regime 2	6.00%	94.00%

Table 2: We show below the estimated volatility matrix for the probability processes when there are 2 regimes with the two risky assets described in Section 4 in the specification used to assess the accuracy of the approximate solution presented by this paper. The estimation procedure is explained in Section 4 and it uses data from January, 1953 to December, 2011. The table also reports the ergodic (unconditional) probabilities and the average duration of each regime.

	Z_1	Z_2	Ergodic Probabilities	Average Duration
Regime 1	-0.0439	0.0302	34.8%	9 months
Regime 2	0.0439	-0.0302	65.2%	17 months

Table 3: We present the optimal portfolios for the two-state model used to assess the accuracy of the approximate solution presented by this paper. The investor has $\gamma = 5$. The portfolios are unconstrained (*i.e.*, short-selling is allowed) as given by our approximate solution. We report the results for different horizons. In the right-most column, we show the monthly wealth equivalent loss due to the approximate solution, when compared to the optimal solution obtained through simulation. The next period regime can be known or unknown. In the latter case, we show results using ergodic and equal probabilities for the next period regime.

Horizon	**Dynamic Portfolio Strategy**			**Monthly Wealth**
(months)	**Stocks**	**Bonds**	**Risk-Free**	**Equivalent Loss**
Known Next Period Regime 1				
1	-16.3%	125.5%	-9.2%	0.00%
6	-14.2%	122.9%	-8.7%	0.00%
12	-14.0%	122.7%	-8.7%	0.00%
60	-14.1%	122.8%	-8.7%	0.00%
120	-14.1%	122.8%	-8.7%	0.00%
Known Next Period Regime 2				
1	176.8%	-187.5%	110.6%	0.00%
6	173.3%	-182.5%	109.2%	0.00%
12	172.5%	-181.5%	108.9%	0.00%
60	172.3%	-181.2%	108.9%	0.00%
120	172.4%	-181.3%	108.9%	0.01%
Unknown Next Period Regime: Ergodic Probabilities				
1	57.7%	35.7%	6.6%	0.00%
6	57.0%	36.7%	6.4%	0.00%
12	56.7%	37.1%	6.3%	0.00%
60	56.5%	37.2%	6.3%	0.00%
120	56.5%	37.2%	6.3%	0.00%
Unknown Next Period Regime: Equal Probabilities				
1	30.4%	75.3%	-5.6%	0.00%
6	30.5%	75.2%	-5.6%	0.00%
12	30.3%	75.4%	-5.7%	0.00%
60	30.2%	75.5%	-5.7%	0.00%
120	30.2%	75.6%	-5.7%	0.00%

Table 4: We report in Panel A the parameters and the optimal weights for the single state model (for $\gamma = 5$). Panel B presents the parameters for the four-state model. Both do not allow for predictability. The correlation matrices present the volatilities in their diagonals. All data is monthly.

Panel A: Single State Model		Large Caps	Small Caps	LT Bonds
	Mean Returns	0.89%	1.13%	0.53%
	Correlation Matrix - Large Caps	0.0420	0.7431	0.1136
	Small Caps	0.7431	0.0605	-0.0030
	LT Bonds	0.1136	-0.0030	0.0213
	Optimal Weights	18.4%	29.1%	40.1%
Panel B: Four State Model		Large Caps	Small Caps	LT Bonds
	Mean Returns - Regime 1 (Crash)	-2.38%	-3.25%	0.59%
	Regime 2 (Slow Growth)	1.14%	0.86%	0.38%
	Regime 3 (Bull)	1.40%	2.49%	0.25%
	Regime 4 (Recovery)	3.91%	6.42%	1.09%
	Correlation Matrix - *Regime 1 (Crash)*			
	Large Caps	0.0516	0.8302	-0.0267
	Small Caps	0.8302	0.0801	0.0061
	LT Bonds	-0.0267	0.0061	0.0255
	Correlation Matrix - *Regime 2 (Slow Growth)*			
	Large Caps	0.0295	0.7146	0.0813
	Small Caps	0.7146	0.0366	-0.0447
	LT Bonds	0.0813	-0.0447	0.0174
	Correlation Matrix - *Regime 3 (Bull)*			
	Large Caps	0.0320	0.7370	0.3559
	Small Caps	0.7370	0.0424	0.0875
	LT Bonds	0.3559	0.0875	0.0052
	Correlation Matrix - *Regime 4 (Recovery)*			
	Large Caps	0.0373	0.1192	0.3303
	Small Caps	0.1192	0.0516	-0.1274
	LT Bonds	0.3303	-0.1274	0.0305
Transition Probabilities	Regime 1	Regime 2	Regime 3	Regime 4
Regime 1 (Crash)	66.11%	0.01%	0.71%	33.17%
Regime 2 (Slow Growth)	6.15%	93.85%	0.00%	0.00%
Regime 3 (Bull)	4.74%	6.26%	88.78%	0.23%
Regime 4 (Recovery)	22.06%	14.19%	7.13%	56.61%

Table 5: We show below the estimated volatility matrix for the probability processes when there are 4 regimes with the three risky assets described in the text and no predictability. The estimation procedure is explained in Section 4 and it uses data from January, 1953 to December, 2011. The table also reports the ergodic (unconditional) probabilities and the average duration of each regime.

	Z_1	Z_2	Z_3	Ergodic Probabilities	Average Duration
Regime 1	-0.0605	-0.0213	0.0025	21.5%	3 months
Regime 2	0.0441	-0.0028	-0.0146	50.2%	16 months
Regime 3	0.0104	0.0112	0.0027	11.8%	9 months
Regime 4	0.0060	0.0129	0.0094	16.5%	2 months

Table 6: We report below the parameters for the four state model with predictability. The correlation matrices present the volatilities in their diagonals. We also present the mean expected returns for the assets under each regime considering that the predictor is equal to its mean value in the respective regime. All data is monthly.

Four State Model with Predictability		Large Caps	Small Caps	LT Bonds
Matrix $\mathbb{A}$ Transposed - Regime 1 (Crash)		-2.75%	-6.41%	1.26%
Regime 2 (Slow Growth)		1.02%	-0.45%	0.69%
Regime 3 (Bull)		0.83%	0.51%	0.30%
Regime 4 (Recovery)		1.88%	4.11%	0.42%
Vector $\mathbb{B}$ Transposed		0.13%	0.62%	0.02%
Mean Returns - Regime 1 (Crash)		-2.61%	-5.73%	1.28%
Regime 2 (Slow Growth)		1.18%	0.30%	0.72%
Regime 3 (Bull)		0.98%	1.20%	0.32%
Regime 4 (Recovery)		2.00%	4.71%	0.44%
Correlation Matrix - *Regime 1 (Crash)*				
Large Caps		0.0623	0.7934	0.1915
Small Caps		0.7934	0.0759	0.1924
LT Bonds		0.1915	0.1924	0.0297
Correlation Matrix - *Regime 2 (Slow Growth)*				
Large Caps		0.0364	0.8069	0.4649
Small Caps		0.8069	0.0410	0.2847
LT Bonds		0.4649	0.2847	0.0249
Correlation Matrix - *Regime 3 (Bull)*				
Large Caps		0.0295	0.7150	-0.1656
Small Caps		0.7150	0.0389	-0.1811
LT Bonds		-0.1656	-0.1811	0.0136
Correlation Matrix - *Regime 4 (Recovery)*				
Large Caps		0.0437	0.6099	0.1236
Small Caps		0.6099	0.0690	-0.0240
LT Bonds		0.1236	-0.0240	0.0250
Transition Probabilities	Regime 1	Regime 2	Regime 3	Regime 4
Regime 1 (Crash)	47.32%	0.01%	0.95%	51.72%
Regime 2 (Slow Growth)	4.87%	95.13%	0.00%	0.00%
Regime 3 (Bull)	4.19%	0.42%	95.30%	0.09%
Regime 4 (Recovery)	11.23%	3.36%	9.52%	75.89%
Predictor's Drifts D_p	0.0545	-0.0036	-0.0023	-0.0236
Predictor's Volatility Vector σ_p	-0.0188	-0.0048	0.0015	0.0226

Table 7: We show below the estimated volatility matrix for the probability processes when there are 4 regimes with the three risky assets described in the text and predictability. The estimation procedure is explained in Section 4 and it uses data from January, 1953 to December, 2011. The table also reports the ergodic (unconditional) probabilities and the average duration of each regime.

	Z_1	Z_2	Z_3	Z_p	Ergodic Prob.	Average Duration
Regime 1	-0.0261	-0.0263	0.0090	0.0003	10.4%	2 months
Regime 2	0.0091	-0.0115	-0.0056	0.0000	19.7%	21 months
Regime 3	0.0341	0.0206	-0.0043	0.0000	47.5%	21 months
Regime 4	-0.0171	0.0172	0.0010	-0.0003	22.4%	4 months

Table 8: We show below the expected total cost of a myopic dynamic strategy when compared to the dynamic strategy with hedging demands as a function of the initial regime running in the economy for three different horizons (given in months). The predictor value is set at its sample average and at this value plus two standard deviations. This cost corresponds to the expected wealth equivalent loss a myopic investor expects to suffer in the beginning of the investment period. Investor's relative risk aversion is set $\gamma = 5$.

Initial State	$p_t = \overline{p}$			$p_t = \overline{p} + 2\delta_{p_t}$		
of Economy	$T = 12$	$T = 60$	$T = 120$	$T = 12$	$T = 60$	$T = 120$
Regime 1	0.04%	0.13%	0.19%	0.11%	1.12%	3.91%
Regime 2	0.01%	0.03%	0.16%	0.01%	0.68%	3.32%
Regime 3	0.01%	0.20%	0.73%	0.04%	1.12%	4.35%
Regime 4	0.03%	0.04%	0.08%	0.10%	0.97%	3.47%
Ergodic Prob.	0.01%	0.12%	0.40%	0.04%	1.00%	3.93%
Equal Prob.	0.01%	0.09%	0.28%	0.05%	0.97%	3.79%

F Figures

Figure 1: Filtered probabilities of being in each of the states. For every point in time, these probabilities are realistically estimated using only the information available before time t. We begin our filter in January, 1952 and after only one year, the probabilities are not sensitive to the starting point. The data range in the plots below is from January, 1962 to December, 2011.

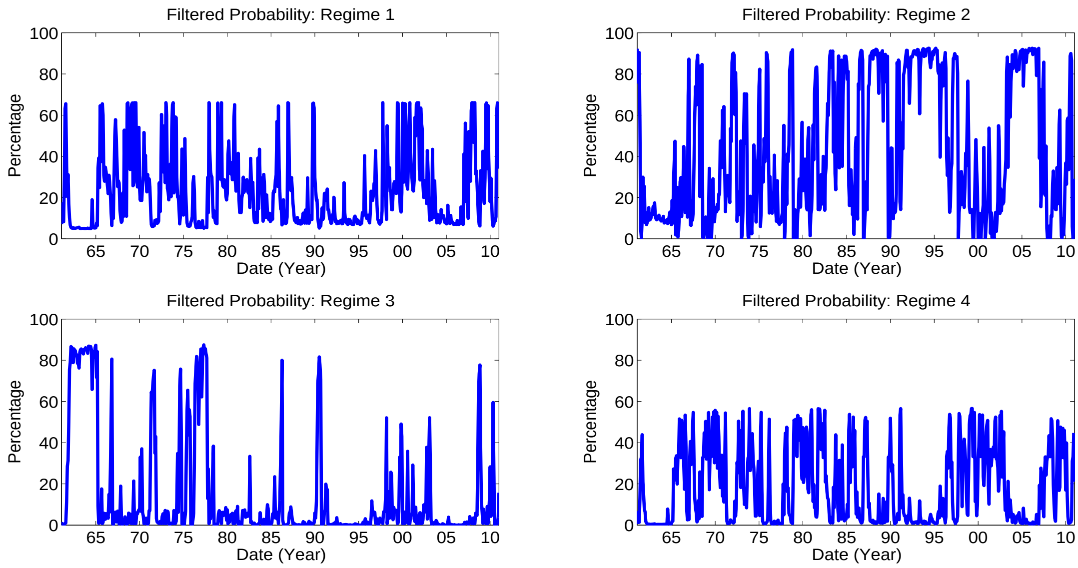

Figure 2: Dynamic strategies for different horizons as given by the model presented in this paper. The horizon is measured in years. The current regime is known. Investor's relative risk aversion is set $\gamma = 5$.

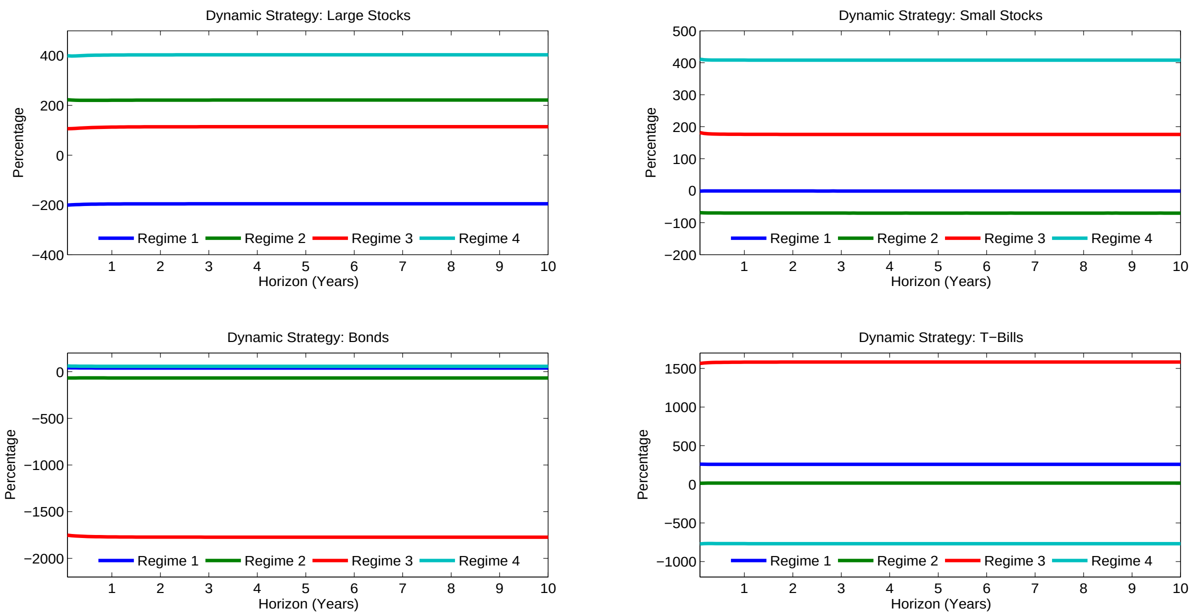

Figure 3: Dynamic strategies for different horizons as given by the model presented in this paper. Short-selling is allowed. The horizon is measured in years. The current regime is unknown and we plot below the strategies when the current probabilities are either the ergodic ones or equal probabilities for each state. Investor's relative risk aversion is set $\gamma = 5$.

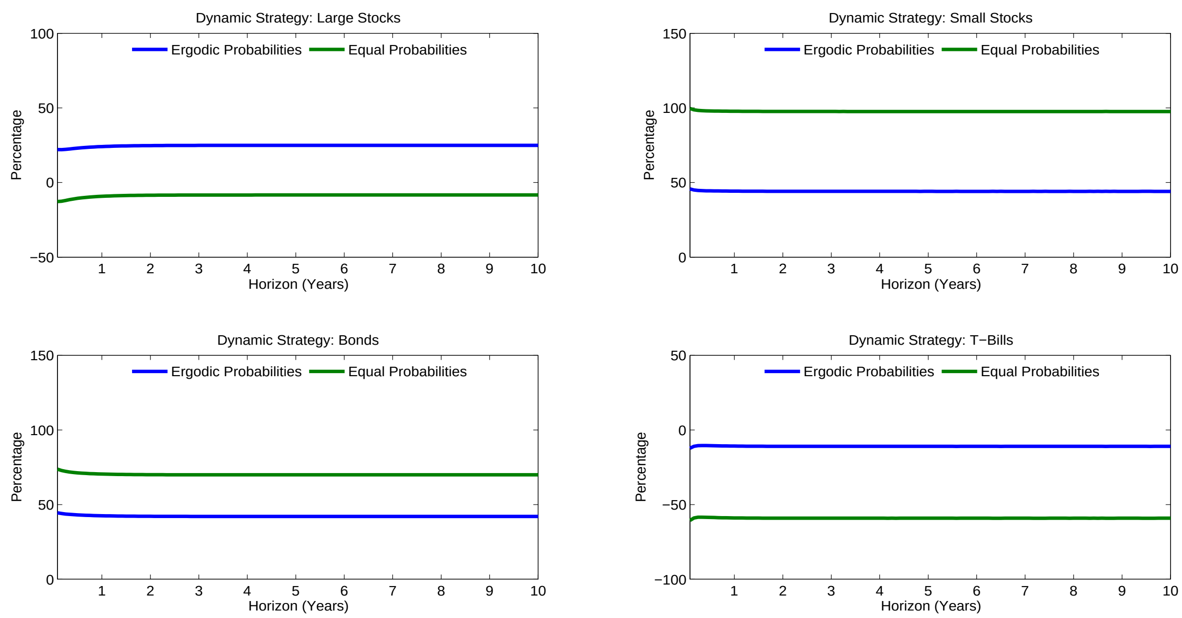

Figure 4: Filtered probabilities of being in each of the states for the problem with predictability. For every point in time, these probabilities are realistically estimated using only the information available before time t. We begin our filter in January, 1952 and after only one year, the probabilities are not sensitive to the starting point. The data range in the plots below is from January, 1962 to December, 2011.

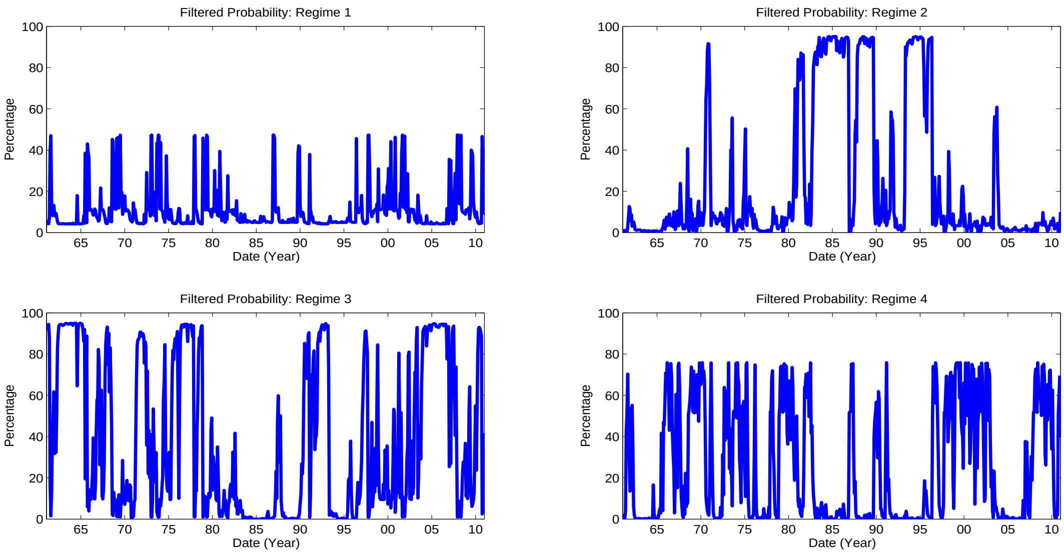

Figure 5: Dynamic strategies for different horizons in the model with predictability. The horizon is measured in years. The current regime is known. Investor's relative risk aversion is set $\gamma = 5$.

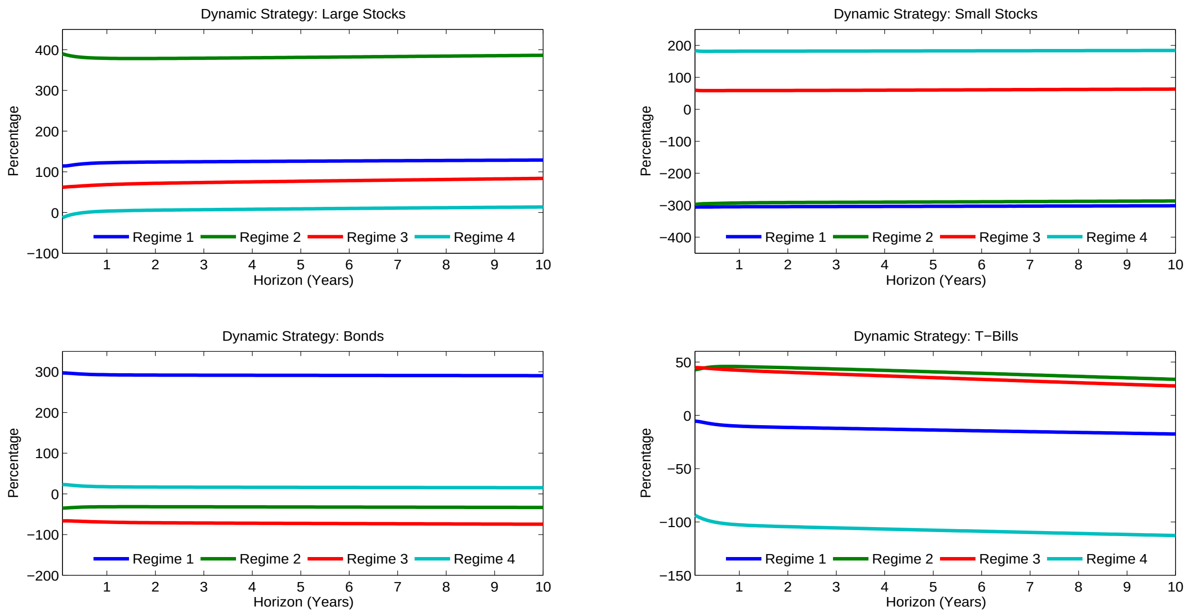

Figure 6: Dynamic strategies for different horizons in the model with predictability. The horizon is measured in years. The current regime is unknown and we plot below the strategies when the current probabilities are either the ergodic ones or equal probabilities for each state. Investor's relative risk aversion is set $\gamma = 5$.

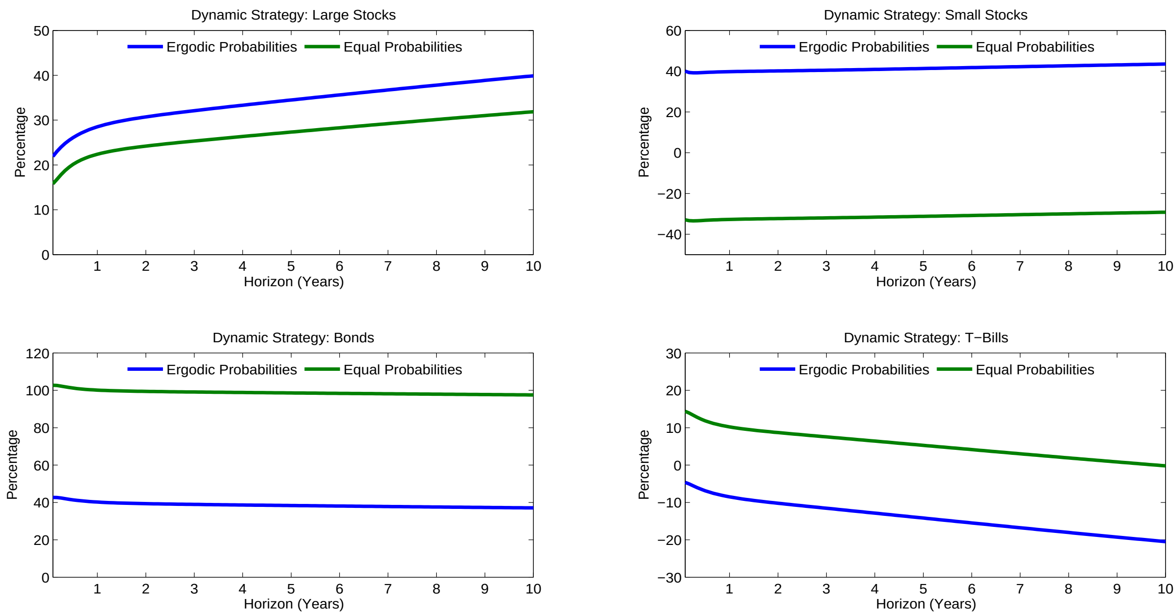

Figure 7: Dynamic strategies for different horizons in the model with predictability. The horizon is measured in years. The current regime is known. Predictor's value is set two standard deviations above its mean. Investor's relative risk aversion is set $\gamma = 5$.

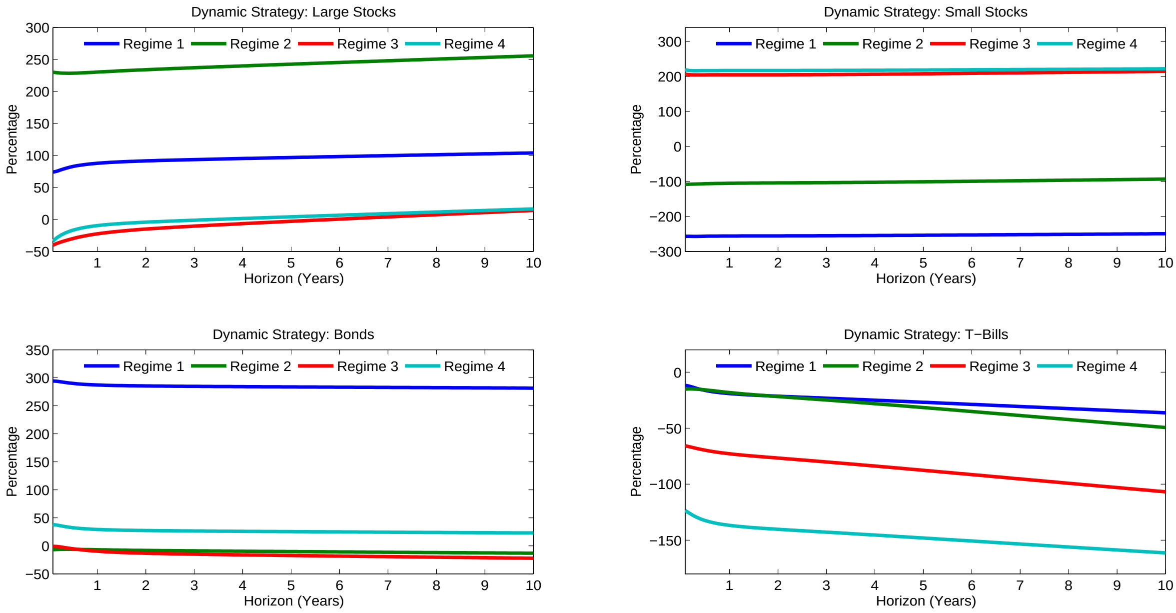

Figure 8: Dynamic strategies for different horizons in the model with predictability. The horizon is measured in years. The current regime is unknown and we plot below the strategies when the current probabilities are either the ergodic ones or equal probabilities for each state. Predictor's value is set two standard deviations above its mean. Investor's relative risk aversion is set $\gamma = 5$.

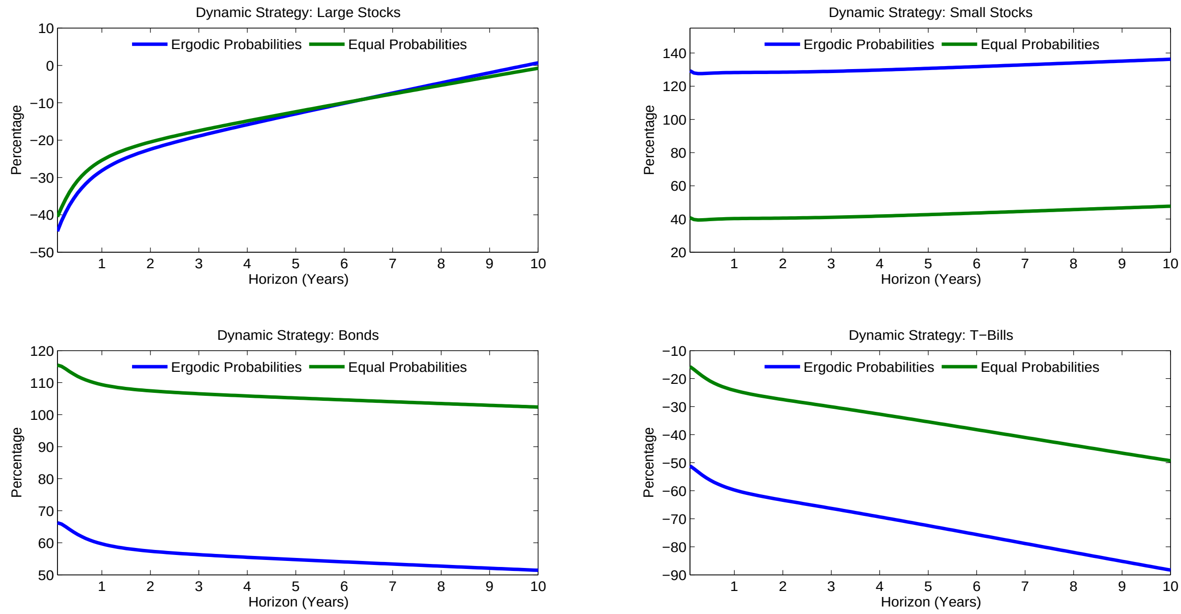

Figure 9: Dynamic strategies for different horizons in the model with predictability. The horizon is measured in years. The current regime is known. Predictor's value is set two standard deviations below its mean. Investor's relative risk aversion is set $\gamma = 5$.

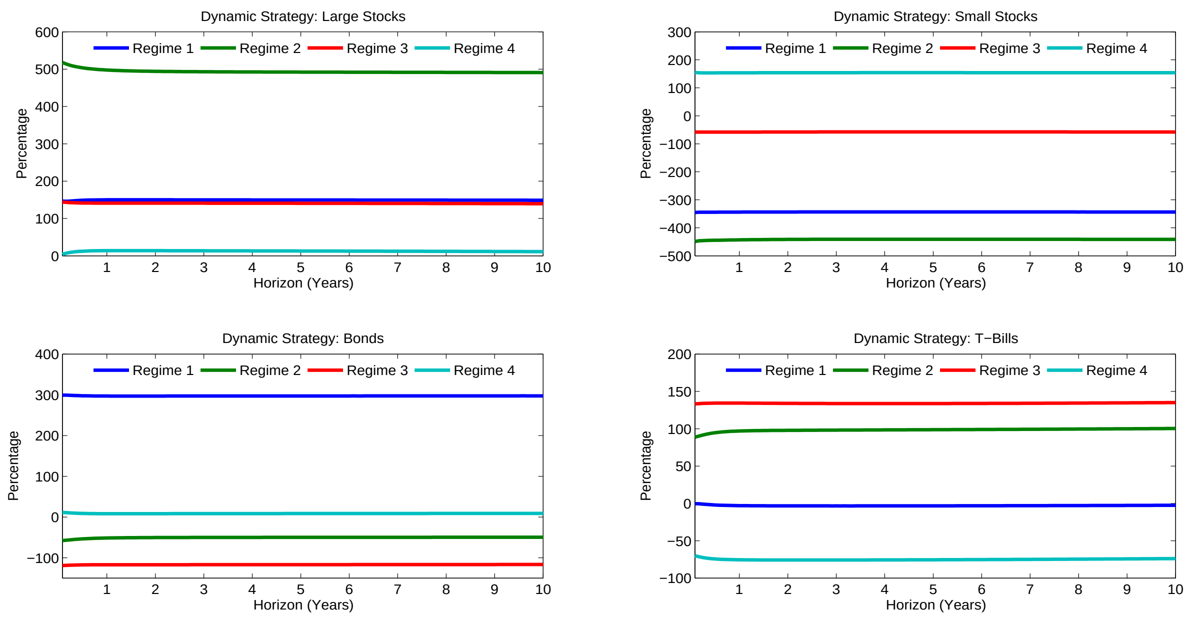

Figure 10: Dynamic strategies for different horizons in the model with predictability. The horizon is measured in years. The current regime is unknown and we plot below the strategies when the current probabilities are either the ergodic ones or equal probabilities for each state. Predictor's value is set two standard deviations below its mean. Investor's relative risk aversion is set $\gamma = 5$.

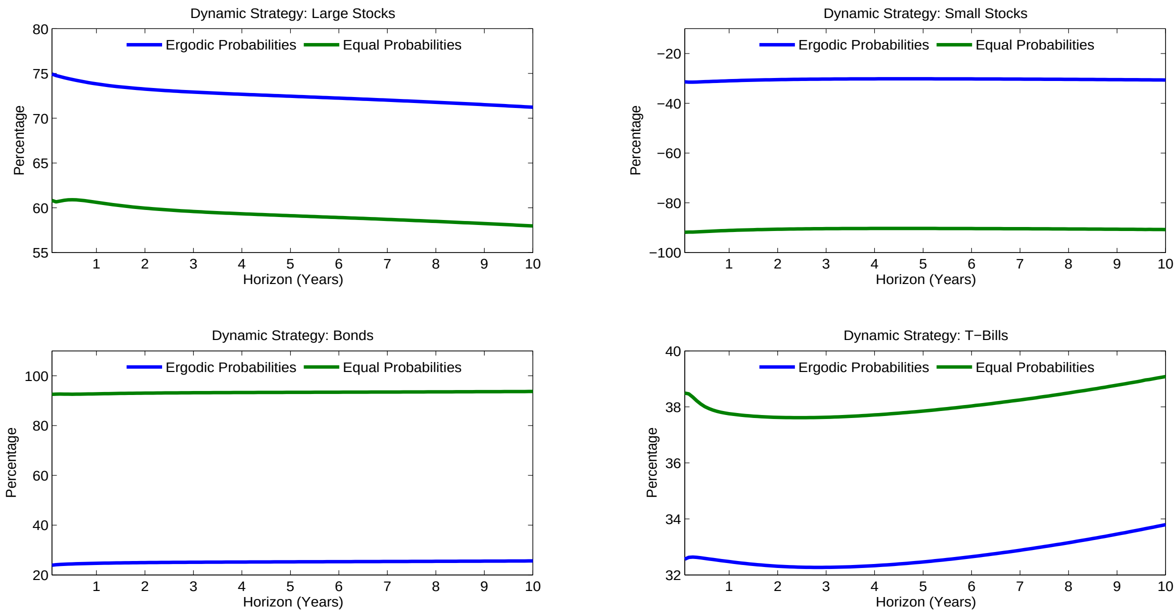

Figure 11: We plot below the consumption strategy for dynamic investors with $\gamma = 5$. In the upper plots, we draw six curves depending on the current regime for horizons from one month to 10 years ($\psi = 0.75$). In the lower plots, we show the consumption strategy followed by a 12-month horizon investor using real-time filtered probabilities (and predictor values) for the period from January, 1980 to December, 2011. The left plots are for the problem with no predictability while the ones in the right, for the model with predictability.