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Chapter 21

Simulation Methods for Optimal Portfolios

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Abstract

This chapter surveys and compares Monte Carlo methods that have been proposed for the computation of optimal portfolio policies. The candidate approaches include the Monte Carlo Malliavin derivative (MCMD) method proposed by Detemple et al. [Detemple, J.B., Garcia, R., Rindisbacher, M. (2003). A Monte-Carlo method for optimal portfolios. *Journal of Finance* 58, 401–446], the Monte Carlo covariation (MCC) method of Cvitanic et al. [Cvitanic, J., Goukasian, L., Zapatero, F. (2003). Monte Carlo computation of optimal portfolio in complete markets. *Journal of Economic Dynamics and Control* 27, 971–986], the Monte Carlo regression (MCR) method of Brandt et al. [Brandt, M.W., Goyal, A., Santa-Clara, P., Stroud, J.R. (2005). A simulation approach to dynamic portfolio choice with an application to learning about return predictability. *Review of Financial Studies* 18, 831–873] and Monte Carlo finite difference (MCFD) methods. The asymptotic properties of the various portfolio estimators obtained are described. A numerical illustration of the convergence behavior of these estimators is provided in the context of a dynamic portfolio choice problem with exact solution. MCMD is shown to dominate other approaches.

1 Introduction

The optimal allocation of wealth among various assets is an important issue that has been of long-standing interest both for academics and practitioners. The workhorse models in the field, have been, for almost 50 years, based on the mean–variance analysis developed by Markowitz (1952). This simple framework brought to light the fundamental notion of a mean–variance trade-off associated with the choice of different securities or portfolios. This popular notion and the associated portfolio rules remain, to this day, at the core of decisions taken and practical recommendations formulated by investment firms and financial advisors.

Yet, mean–variance portfolio rules have been known to be flawed for over 3 decades. In a seminal contribution, Merton (1971) identified the main problem, their failure to account for stochastic shifts in the investment opportunity set (i.e. in means and variances).[1] While of little consequence for very short term investors, this failure proves important for economic units with long horizons. Indeed, only a very particular class of long term investors, namely those with unit relative risk aversion (logarithmic utility), will find it optimal to behave as short term mean–variance optimizers. Generic long term investors follow amended portfolio rules that include intertemporal hedging terms, in addition to mean–variance components. The benefit of those hedging terms is intuitively clear: in a stochastically changing environment it pays to take intertemporal links into account and hedge against variations in means and variances.

Numerical methods for computing these hedging terms and the associated optimal portfolios have notably lagged behind. Much of the earlier literature has indeed searched for closed form solutions in the context of simple parametric models, with limited assets and state variables and simple dynamics. The earliest attempt to numerically solve a nontrivial portfolio choice problem can perhaps be attributed to Brennan et al. (1997), who examine a model with 3 assets and 4 state variables. Their approach uses numerical methods for partial differential equations (PDEs) and is based on the dynamic programming characterization of the optimal solution developed by Merton. Their study reveals the importance of the dynamic portfolio choice problem and highlights some of the difficulties that need to be overcome.

Rapid developments have followed. Simulation methods were first proposed by Detemple et al. (2003), who exploit a portfolio formula based on Malliavin calculus derived by Ocone and Karatzas (1991) for Itô price processes. Their basic method, labeled Monte Carlo Malliavin derivative (MCMD), involves the simulation of state variables and Malliavin derivatives to compute the expectations arising in the portfolio components. A variation of the method applies a change of variables (a Doss transformation) and simulates the transformed variables and their Malliavin derivatives to compute the relevant expressions. This Monte Carlo Malliavin derivative method with Doss transformation (MCMD-Doss) is proposed and studied in Detemple et al. (2003, 2005a, 2005c). An alternative, suggested by Cvitanic et al. (2003), uses an approximation of the optimal portfolio rule, based on the covariation between wealth and the underlying Brownian motions, as the basis for Monte

[1] See also Breeden (1979).

Carlo simulation. This simple approach, called the Monte Carlo covariation (MCC) method, is easy to implement as it only involves the simulation of the primitive state variables. Another approximation, that combines dynamic programming with regressions and simulations, is advocated by Brandt et al. (2005). It relies on an approximation of the optimality conditions for the portfolio and uses a regression-simulation method to evaluate conditional expectations in the coefficients of the approximate portfolio conditions. This Monte Carlo regression (MCR) scheme is reminiscent of the regression method developed by Longstaff and Schwartz (2001) for American options' valuation. Monte Carlo finite difference (MCFD) methods complete the list of simulation approaches that have been proposed to date for optimal portfolio calculations. This approach exploits the link between Malliavin derivatives and tangent processes (loosely speaking derivatives with respect to initial conditions) and evaluates the relevant derivatives using finite differences. MCFD approaches are described in Detemple et al. (2005d) and evaluated along with MCMD and MCC in the context of risk management problems.

This chapter surveys the recent literature on simulation methods for optimal portfolios. The various methods, informally described above, are presented in details and discussed. A numerical study is performed to evaluate their relative performances. MCMD is shown to dominate other candidate approaches.

Section 2 outlines the consumption-portfolio choice problem in a setting with complete markets and von Neumann–Morgenstern preferences and presents several representations formulas for its solution. Simulation methods for optimal portfolio calculations are reviewed in Section 3. Asymptotic properties of portfolio estimators are examined in Section 4 and a numerical study of the convergence behavior of the various methods is conducted in Section 5. Concluding remarks and avenues for future work are outlined in the last section. Appendix A presents elementary rules of Malliavin calculus that are needed to derive formulas underlying some of the methods. Proofs are collected in Appendix B.

2 The consumption-portfolio choice problem

We formulate a continuous time consumption-portfolio choice model in the tradition of Merton (1971). A finitely-lived investor operates in a frictionless economy in which asset prices and state variables follow a joint diffusion process. The investor's planning horizon is $[0, T]$.

2.1 The financial market

The financial market has d risky assets (stocks) and 1 locally riskless

$$\mathrm{d}S_{it} = S_{it}\big[\big(\mu_i(t, Y_t) - \delta_i(t, Y_t)\big)\,\mathrm{d}t + \sigma_i(t, Y_t)'\,\mathrm{d}W_t\big]; \quad S_{i0} \text{ given,} \tag{2.1}$$

where μ_i represents the return's drift, δ_i the dividend yield and σ_i' the $1 \times d$ vector of volatility coefficients. The coefficients of (2.1) depend on a $k \times 1$ vector of state variables $Y = (Y_1, \ldots, Y_k)'$. The interest rate on the riskless asset, $r(t, Y_t)$, also depends on the state variables. To simplify notation we will write μ_t for the $d \times 1$ vector of expected risky asset returns at date t, δ_t for the $d \times 1$ vector of dividend yields, σ_t for the $d \times d$ matrix of return volatilities and r_t for the interest rate. We assume that σ is invertible at all times (i.e. the market is complete).

The price system (2.1) induces a unique d-dimensional vector of market prices of risk $\theta_t = (\theta_{1t}, \ldots, \theta_{dt})'$ defined by $\theta_t \equiv \sigma_t^{-1}(\mu_t - r_t 1_d)$ where $1_d = (1, \ldots, 1)'$ is the d-dimensional vector of ones. The market prices of risk represent the premia implicitly assigned by the financial market to the sources of uncertainty (the Brownian motions) affecting the economy. The state price density (SPD), $\xi_v \equiv \exp(-\int_0^v (r_s + \frac{1}{2}\theta_s'\theta_s)\,\mathrm{d}s - \int_0^v \theta_s'\,\mathrm{d}W_s)$, is the stochastic discount factor that matters to find the value at date 0 of cash flows received at $v \geqslant 0$. The relative state price density (RSPD), $\xi_{t,v} \equiv \exp(-\int_t^v (r_s + \frac{1}{2}\theta_s'\theta_s)\,\mathrm{d}s - \int_t^v \theta_s'\,\mathrm{d}W_s) = \xi_v/\xi_t$, is the stochastic discount factor that matters to find the value at date t of cash flows received at $v \geqslant t$.

2.2 State variables

The state variables $Y = (Y_1, \ldots, Y_k)'$ affect the coefficients of asset returns and the riskfree rate (i.e. the opportunity set). The list of state variables can include the market prices of risk and the interest rate (e.g. $Y_1 = r$ and $Y_j = \theta_j$, $j = 2, \ldots, d+1$). Additional variables that could be relevant include dividend–price ratios, measures of firm sizes and measures of sales or revenues. State variables are assumed to evolve according to

$$\mathrm{d}Y_t = \mu^Y(t, Y_t)\,\mathrm{d}t + \sigma^Y(t, Y_t)\,\mathrm{d}W_t; \quad Y_0 \text{ given,} \tag{2.2}$$

where $\mu^Y(t, Y_t)$ is the $k \times 1$ vector of drift coefficients and $\sigma^Y(t, Y_t)$ is a $k \times d$ matrix of volatility coefficients.

2.3 Consumption, portfolios and wealth

The investor under consideration consumes and allocates his/her wealth among the different assets available. Let X_t be wealth at date t. Consumption is c_t and π_t is the $d \times 1$ vector of wealth proportions invested in the risky assets (thus $1 - \pi_t' 1_d$ is the proportion invested in the riskless asset). The evolution of wealth is governed by the stochastic differential equation

$$\mathrm{d}X_t = (X_t r_t - c_t)\,\mathrm{d}t + X_t \pi_t'\big[(\mu_t - r_t 1_d)\,\mathrm{d}t + \sigma_t\,\mathrm{d}W_t\big] \tag{2.3}$$

subject to the initial condition $X_0 = x$.

2.4 Preferences

Preferences are assumed to have the time-separable von Neumann–Morgenstern representation. A consumption-terminal wealth plan (c, X_T) is ranked according to the criterion

$$\mathbf{E}\left[\int_0^T u(c_v, v)\,\mathrm{d}v + U(X_T, T)\right] \tag{2.4}$$

where the utility functions $u:[d_1,\infty)\times[0,T]\to\mathbb{R}$ and $U:[d_2,\infty)\to\mathbb{R}$ are strictly increasing, strictly concave and differentiable over their respective domains. We also assume that the limiting conditions $\lim_{c\to d_1} u'(c,t) = \lim_{X\to d_2} U'(X,T) = \infty$ and $\lim_{c\to\infty} u'(c,t) = \lim_{X\to\infty} U'(X,T) = 0$ hold for all $t\in[0,T]$. If domains include $\mathbb{R}_+\times[0,T]$ (i.e. $d_1, d_2 \leqslant 0$) no further restrictions are imposed. If $[d_1,\infty)$ is a proper subset of $\mathbb{R}_+$ (i.e. $d_1 > 0$) we extend the function u to $\mathbb{R}_+\times[0,T]$ by setting $u(c,t) = -\infty$ for $c\in\mathbb{R}_+\backslash[d_1,\infty)$ and for all $t\in[0,T]$. We proceed in the same manner to extend U if $[d_2,\infty)$ is a proper subset of $\mathbb{R}_+$.

This class of utility functions includes the HARA specification

$$u(c,t) = \frac{1}{1-R}(c+A)^{1-R},$$

where $R > 0$. If A is positive the utility function $u(c,t)$ is defined over the domain $[d_1,\infty) = [-A,\infty)$ and satisfies all the required conditions. If $A < 0$ the function has the required properties over the subset $[d_1,\infty) = [-A,\infty)\subset\mathbb{R}_+$. The function is then extended by setting $u(c,t) = -\infty$ for $c \leqslant d_1$. This particular HARA specification corresponds to a model with subsistence consumption $-A$.

Under these assumptions the respective inverses $I:\mathbb{R}_+\times[0,T]\to[d_1,\infty)$ and $J:\mathbb{R}_+\to[d_2,\infty)$ of the marginal utility functions $u'(c,t)$ and $U'(X,T)$ exist and are unique. They are also strictly decreasing with limiting values $\lim_{y\to 0} I(y,t) = \lim_{y\to 0} J(y,T) = \infty$ and $\lim_{y\to\infty} I(y,t) = d_1$, $\lim_{y\to\infty} J(y,T) = d_2$.

2.5 The dynamic consumption-portfolio choice problem

The investor seeks to maximize expected utility

$$\max_{(c,\pi)} \mathbf{E}\left[\int_0^T u(c_v, v)\,\mathrm{d}v + U(X_T, T)\right] \tag{2.5}$$

subject to the following constraints

$$\mathrm{d}X_t = (r_t X_t - c_t)\,\mathrm{d}t + X_t \pi_t'\big[(\mu_t - r_t 1_d)\,\mathrm{d}t + \sigma_t\,\mathrm{d}W_t\big];\quad X_0 = x, \tag{2.6}$$

$$c_t \geqslant 0, \qquad X_T \geqslant 0 \tag{2.7}$$

for all $t \in [0, T]$. The first constraint, (2.6), describes the evolution of wealth given a consumption-portfolio policy (c, π). The next one (2.7) captures the physical restriction that consumption and bequest cannot become negative. This constraint ensures that wealth, that is the present value of future consumption, cannot become negative.

2.6 Optimal consumption, portfolio and wealth

Standard results of Pliska (1986), Karatzas et al. (1987) and Cox and Huang (1989) (see Karatzas and Shreve, 1998) can be invoked to show that the optimal consumption policy is

$$c_t^* = I(y^* \xi_t, t)^+ = \max\{I(y^* \xi_t, t), 0\}, \tag{2.8}$$

$$X_T^* = J(y^* \xi_T, T)^+ = \max\{J(y^* \xi_T, T), 0\} \tag{2.9}$$

where the constant y^* is the unique solution of the static budget constraint

$$\mathbf{E}\left[\int_0^T \xi_v I(y\xi_v, v)^+ \, \mathrm{d}v + \xi_T J(y\xi_T, T)^+\right] = x \tag{2.10}$$

with $x \geqslant \max\{\mathbf{E}[\int_0^T \xi_v d_1 \, \mathrm{d}v + \xi_T d_2], 0\}$.

The resulting wealth process is the present value of optimal future consumption and is therefore given by

$$X_t^* = \mathbf{E}_t\left[\int_t^T \xi_{t,v} I(y^* \xi_v, v)^+ \, \mathrm{d}v + \xi_{t,T} J(y^* \xi_T, T)^+\right] \equiv \mathbf{E}_t[F_{t,T}] \tag{2.11}$$

for $t \in [0, T]$, where $F_{t,T} \equiv \int_t^T \xi_{t,v} I(y^* \xi_v, v)^+ \, \mathrm{d}v + \xi_{t,T} J(y^* \xi_T, T)^+$. The associated optimal portfolio can be expressed as

$$X_t^* \pi_t^* = X_t^* (\sigma_t')^{-1} \theta_t + \xi_t^{-1} (\sigma_t')^{-1} \phi_t, \tag{2.12}$$

where ϕ is the predictable process in the representation of the martingale $M_t = \mathbf{E}_t[F] - \mathbf{E}[F]$ with $F \equiv F_{0,T} = \int_0^T \xi_t c_t^* \, \mathrm{d}t + \xi_T X_T^*$.

2.7 The optimal portfolio: an explicit formula

To find a more explicit expression for the optimal portfolio it remains to identify the process ϕ in (2.12). The Clark–Ocone formula (see Appendix A) becomes instrumental for that purpose: it identifies the integrand in the representation of the martingale M and enables us to express the optimal portfolio in terms of the parameters of the model (i.e. the structure of F). Ocone and Karatzas (1991) establish this portfolio formula for general models with Itô price processes. The specialization to diffusions can be found in Detemple et al. (2003).

Applying the Clark–Ocone formula and using the rules of Malliavin calculus shows that

$$\phi_t = \mathbf{E}_t[\mathcal{D}_t F]$$

where

$$\mathcal{D}_t F = \mathbf{E}_t\left[\int_t^T Z_1(y^*\xi_v, v)\mathcal{D}_t\xi_v \,\mathrm{d}v + Z_2(y^*\xi_T, T)\mathcal{D}_t\xi_T\right] \tag{2.13}$$

with

$$\begin{aligned} Z_1(y^*\xi_v, v) &= I(y^*\xi_v, v)^+ + y^*\xi_v I'(y^*\xi_v, v)1_{\{I(y^*\xi_v, v)\geqslant 0\}} \\ &= c_v^*\left(1 - \frac{1}{R_u(c_v^*, v)}\right), \end{aligned} \tag{2.14}$$

$$\begin{aligned} Z_2(y^*\xi_T, T) &= J(y^*\xi_T, T)^+ + y^*\xi_T J'(y^*\xi_T, T)1_{\{J(y^*\xi_T, T)\geqslant 0\}} \\ &= X_T^*\left(1 - \frac{1}{R_U(X_T^*, T)}\right). \end{aligned} \tag{2.15}$$

In these expressions $I'(y^*\xi_v, v)$, $J'(y^*\xi_T, T)$ are the derivatives with respect to the first argument $y^*\xi$ of the inverse marginal utility functions and $R_u(x, v) = -u_{cc}(x, v)x/u_c(x, v)$, $R_U(x, T) = -U_{XX}(x, T)x/U_X(x, T)$ are relative risk aversion coefficients.

From the definition of the stochastic discount factor ξ in Section 2.1 we obtain

$$\mathcal{D}_t\xi_v \equiv -\xi_v\left(\int_t^v (\mathcal{D}_t r_s + \theta_s'\mathcal{D}_t\theta_s)\,\mathrm{d}s + \int_t^v \mathrm{d}W_s' \cdot \mathcal{D}_t\theta_s + \theta_t'\right).$$

The chain rule of Malliavin calculus then gives $\mathcal{D}_t\xi_v = -\xi_v(H_{t,v}' + \theta_t')$ with

$$H_{t,v}' = \int_t^v \big(\partial r(s, Y_s) + \theta_s'\partial\theta(s, Y_s)\big)\mathcal{D}_t Y_s\,\mathrm{d}s + \int_t^v \mathrm{d}W_s' \cdot \partial\theta(s, Y_s)\mathcal{D}_t Y_s \tag{2.16}$$

and where $\mathcal{D}_t Y_s$ satisfies the stochastic differential equation

$$\begin{aligned} \mathrm{d}\mathcal{D}_t Y_s &= \left[\partial\mu^Y(s, Y_s)\,\mathrm{d}s + \sum_{j=1}^d \partial\sigma_j^Y(s, Y_s)\,\mathrm{d}W_s^j\right]\mathcal{D}_t Y_s; \\ \mathcal{D}_t Y_t &= \sigma^Y(t, Y_t). \end{aligned} \tag{2.17}$$

In this expression the notation $\partial f(Y)$ stands for the $p\times k$-dimensional Jacobian matrix of a p-dimensional vector function f with respect to the k-dimensional vector Y. Substituting (2.13)–(2.17) back in (2.12), collecting terms and simplifying leads to our explicit portfolio formula (for details see the proof of Proposition 1 in Appendix B). Our next proposition summarizes the results

Proposition 1. *Consider the dynamic consumption-portfolio problem (2.6)–(2.7). The optimal consumption policy is*

$$c_v^* = I(y^*\xi_v, v)^+, \qquad X_T^* = J(y^*\xi_T, T)^+. \tag{2.18}$$

The optimal portfolio policy has the decomposition $X_t^*\pi_t^* = X_t^*[\pi_{1t}^* + \pi_{2t}^*]$ *where* π_{1t}^* *is the mean–variance demand and* π_{2t}^* *the intertemporal hedging demand. The two components are*

$$X_t^*\pi_{1t}^* = -\mathbf{E}_t[D_{t,T}](\sigma_t')^{-1}\theta_t, \tag{2.19}$$

$$X_t^*\pi_{2t}^* = -(\sigma_t')^{-1}\mathbf{E}_t[G_{t,T}], \tag{2.20}$$

where

$$D_{t,T} \equiv \int_t^T \xi_{t,v}(y^*\xi_v)I'(y^*\xi_v, v)1_{\{I(y^*\xi_v,v)\geqslant 0\}}\,\mathrm{d}v + \xi_{t,T}(y^*\xi_T)J'(y^*\xi_T, T)1_{\{J(y^*\xi_T,T)\geqslant 0\}}, \tag{2.21}$$

$$G_{t,T} \equiv \int_t^T \xi_{t,v}Z_1(y^*\xi_v, v)H_{t,v}\,\mathrm{d}v + \xi_{t,T}Z_2(y^*\xi_T, T)H_{t,T} \tag{2.22}$$

and where $Z_1(y^*\xi_v, v)$ *and* $Z_2(y^*\xi_T, T)$ *are given in (2.14)–(2.15), the random variable* $H_{t,v}$ *is defined in (2.16) and the Malliavin derivative of the state variables,* $\mathcal{D}_tY_s$*, satisfies the stochastic differential equation (2.17). The multiplier* y^* *solves the nonlinear equation (2.10). Optimal wealth is* $X_t^* = \mathbf{E}_t[F_{t,T}]$.

The portfolio decomposition described in this proposition reflects two investment motives. The first one, which underlies the mean–variance demand π_1, is driven by the trade-off between risk and return embedded in asset returns. This motive, originally identified by Markowitz (1952), has played an important role in portfolio theory and remains at the core of practical implementations. The second one, underlying the demand component π_2, is a hedging motive prompted by stochastic fluctuations in the opportunity set. This intertemporal motive, identified by Merton (1971), is a fundamental aspect of optimal dynamic portfolio policies whose implementation has become a focus of current practice.

For later developments we record the special case of constant relative risk aversion in the following corollary:

Corollary 1. *Suppose that the investor exhibits constant relative risk aversion* R *and has subjective discount factor* $\eta_t \equiv \exp(-\beta t)$ *where* β *is a constant rate.*

The optimal consumption policy is given by $c_v^* = (y^* \xi_v / \eta_v)^{-1/R}$ *and* $X_T^* = (y^* \xi_T / \eta_T)^{-1/R}$. *The optimal portfolio is* $X_t^* \pi_t^* = X_t^* [\pi_{1t}^* + \pi_{2t}^*]$ *where*

$$X_t^* \pi_{1t}^* = \frac{X_t^*}{R} (\sigma_t')^{-1} \theta_t, \tag{2.23}$$

$$X_t^* \pi_{2t}^* = -X_t^* \rho (\sigma_t')^{-1} \frac{\mathbf{E}_t[\int_t^T \xi_{t,v}^\rho \eta_{t,v}^{1/R} H_{t,v} \, dv + \xi_{t,T}^\rho \eta_{t,T}^{1/R} H_{t,T}]}{\mathbf{E}_t[\int_t^T \xi_{t,v}^\rho \eta_{t,v}^{1/R} \, dv + \xi_{t,T}^\rho \eta_{t,T}^{1/R}]} \tag{2.24}$$

with $\rho = 1 - 1/R$ *and* $H_{t,v}$ *defined in* (2.16).

For general model structures the conditional expectations appearing in the formulas of Proposition 1 and Corollary 1 cannot be calculated in more explicit form. Numerical methods must then be used in order to implement the optimal portfolio policies. The complexity inherent in the random variables $\xi_{t,v}, H_{t,v}$ appearing in the expressions obtained, and in particular their path-dependent nature, naturally suggests the use of Monte Carlo simulation for computation purposes.

2.8 *Malliavin derivative representation and dynamic programming*

The classic approach to the consumption-portfolio choice problem in a Markovian setting was pioneered by Merton (1971) and is based on dynamic programming principles. Let $V(t, X^*, Y)$ be the value function associated with the problem. The optimal consumption and terminal wealth policies and the optimal portfolio are expressed in terms of the derivatives $V_t, V_x, V_y, V_{xx}, V_{xy}, V_{yy}$ of the value function as

$$c_t^* = I\big(V_x(t, X_t^*, Y_t), t\big)^+, \qquad X_T^* = J\big(V_x(T, X_T^*, Y_T), T\big)^+, \tag{2.25}$$

$$X_t^* \pi_t^* = \frac{V_x(t, X_t^*, Y_t)}{-V_{xx}(t, X_t^*, Y_t)} \big(\sigma(t, Y_t)'\big)^{-1} \theta(t, Y_t) + \big(\sigma(t, Y_t)'\big)^{-1} \sigma^Y(t, Y_t)' \frac{V_{yx}(t, X_t^*, Y_t)}{-V_{xx}(t, X_t^*, Y_t)}. \tag{2.26}$$

The value function solves the partial differential equation (PDE)

$$u\big(I(V_x, t)^+, t\big) - V_x I(V_x, t)^+ + V_t + V_y \mu^Y + \frac{1}{2} \text{trace}\{V_{yy} \sigma^Y (\sigma^Y)'\} - \frac{1}{2} V_{xx} \|\psi'\|^2 = 0, \tag{2.27}$$

where $\psi \equiv \frac{V_x}{-V_{xx}} \theta + (\sigma^Y)' \frac{V_{yx}}{-V_{xx}}$, subject to the boundary conditions $V(T, x, y) = U(x, T)$ and $V(t, 0, y) = \int_t^T u(0, s) \, ds + U(0, T)$.

Our next result draws the link between Merton's solution and the probabilistic representation obtained in Proposition 1.

Proposition 2. *The state price density is proportional to the wealth derivative of the value function*

$$y^* \xi_t = V_x(t, X_t^*, Y_t), \quad or \quad \xi_t = \frac{V_x(t, X_t^*, Y_t)}{V_x(0, X_0^*, Y_0)} \tag{2.28}$$

ensuring that the optimal consumption and terminal wealth policies in (2.18) and (2.25) are identical. The scaling factors in the mean–variance and hedging demands (2.19)–(2.20) are given by

$$-\mathbf{E}_t[D_{t,T}] = \frac{V_x(t, X_t^*, Y_t)}{-V_{xx}(t, X_t^*, Y_t)}, \tag{2.29}$$

$$-\mathbf{E}_t[G_{t,T}] = \sigma^Y(t, Y_t)' \frac{V_{xy}(t, X_t^*, Y_t)}{-V_{xx}(t, X_t^*, Y_t)}. \tag{2.30}$$

Formulas (2.18)–(2.22) are alternative representations of the solution as expressed in (2.25)–(2.27).

Proposition 2 shows that our previous expressions for the optimal policies, (2.18)–(2.22), are probabilistic representations of the formulas derived by Merton. These representation are in the spirit of Feynman–Kac as they express the elements of the HJB equation in terms of conditional expectations of random variables. Note, in particular, that $-\mathbf{E}_t[D_{t,T}]$ in (2.29) is simply the probabilistic representation of the coefficient of "absolute risk tolerance" of the indirect utility function V (the value function).[2]

The results in Proposition 2 shed light on the relation between the value function and the fundamentals of the model, namely preferences and the state price density. For instance, it is well known that the hedging motive vanishes (at all times and in all states) if and only if the vector of cross partial derivatives of the value function, V_{xy}, is identically equal to zero. The Malliavin derivative representation in (2.30) and Equation (2.22) show that this condition is satisfied if and only if the processes (r, θ) are deterministic and/or the investor displays myopic behavior ($R_u = R_U = 1$).

2.9 Malliavin derivative and tangent process

For interpretation and computational purposes it is also instructive to rewrite the portfolio policy in terms of the derivative of the state variables with respect to their initial values. This derivative, called the tangent process (or first variation process), is described in Appendix A (see Section A.8).

The tangent process of Y, denoted by $\nabla_{t,y} Y \equiv \{\nabla_{t,y} Y_v : v \in [t, T]\}$, captures the change in the future values of Y following an incremental perturbation of

[2] The coefficient of absolute risk aversion $A(x)$ of a utility function u is $A(x) \equiv R(x)/x$ where $R(x) = -u''(x)x/u'(x)$ is relative risk aversion. Absolute risk tolerance is $1/A(x)$.

the position $Y_t = y$ at time t. In particular, for $v \geqslant t$, $\nabla_{t,y} Y_v$ is the variation in Y_v due to the initial perturbation. The tangent process is easy to characterize when Y solves an SDE. In fact, one can verify that $\nabla_{t,y} Y$ solves the SDE

$$d(\nabla_{t,y} Y_v) = \left(\partial \mu^Y(v, Y_v)\,\mathrm{d}v + \sum_{j=1}^{d} \partial \sigma_j^Y(v, Y_v)\,\mathrm{d}W_v^j \right) \nabla_{t,y} Y_v;$$
$$\nabla_{t,y} Y_t = I_k, \tag{2.31}$$

where I_k is the k-dimensional identity matrix. A comparison of (2.31) with (2.17) shows that the equation for the tangent process differs from the one for the Malliavin derivative only through the initial condition ($\nabla_{t,y} Y_t = I_k$ versus $\mathcal{D}_t Y_t = \sigma^Y(t, Y_t)$). It follows immediately that the relationship

$$\mathcal{D}_t Y_t = \nabla_{t,y} Y_v \sigma^Y(t, Y_t) \tag{2.32}$$

holds. The tangent process can be viewed as a normalized version of the Malliavin derivative. Conversely, the Malliavin derivative is a linear transformation of the tangent process.

Relationship (2.32) between the two notions enables us to rewrite the hedging term (2.20) in the form

$$X_t^* \pi_{2t}^* = -\big(\sigma_t(t, Y_t)'\big)^{-1} \sigma^Y(t, Y_t)' \mathbf{E}_t\big[G_{t,T}(\Phi)\big] \tag{2.33}$$

where

$$G_{t,T}(\Phi) \equiv \int_t^T \xi_{t,v} Z_1(y^* \xi_v, v) \Phi_{t,v}\,\mathrm{d}v + \xi_{t,T} Z_2(y^* \xi_T, T) \Phi_{t,T} \tag{2.34}$$

and

$$\Phi'_{t,v} \equiv \int_t^v \big(\partial r(s, Y_s) + \theta_s' \partial \theta(s, Y_s)\big) \nabla_{t,y} Y_s\,\mathrm{d}s$$
$$+ \int_t^v \mathrm{d}W_s' \cdot \partial \theta(s, Y_s) \nabla_{t,y} Y_s. \tag{2.35}$$

To derive this representation we used $H'_{t,v} = \Phi'_{t,v} \sigma^Y(t, Y_t)$. For computations it is also useful to note that $\Phi_{t,v} = -\nabla_{t,y} \log(\xi_{t,v})$: the functional $\Phi_{t,v}$ is the variation of $-\log(\xi_{t,v})$ for a perturbation in the position of the state variables $Y_t = y$ at time t. Finally, one can write the general representation

$$X_t^* \pi_{2t}^* = -\big(\sigma_t(t, Y_t)'\big)^{-1} \sigma^Y(t, Y_t)' \mathbf{E}_t\big[(\nabla_{t,y} F_{t,T})'\big] \tag{2.36}$$

where the functional $F_{t,T}$ is as defined in (2.11) and $(\nabla_{t,y} F_{t,T})' = G_{t,T}(\Phi)$. A comparison of (2.33) with (2.30) also shows that

$$\frac{V_{xy}(t, X_t^*, Y_t)}{-V_{xx}(t, X_t^*, Y_t)} = -\mathbf{E}_t\left[\int_t^T \xi_{t,v} Z_1(y^* \xi_v, v) \Phi_{t,v}\, \mathrm{d}v + \xi_{t,T} Z_2(y^* \xi_T, T) \Phi_{t,T}\right]. \tag{2.37}$$

This relation captures the intuitive notion that the hedging coefficient is related to the impact of a perturbation in the state variables at date t on the optimal wealth. This effect is precisely the expectation on the right-hand side of (2.37).

3 Simulation methods for portfolio computation

This section reviews various Monte Carlo methods that have been proposed for the computation of asset allocation rules.

3.1 Monte Carlo Malliavin derivatives (Detemple et al., 2003)

This simulation approach, developed by Detemple et al. (2003), is directly based on the formulas described in Section 2.7. Suppose that we are in the general context of Proposition 1 where the multiplier y^* for the static budget constraint cannot be solved explicitly from (2.10). Consider first the case where y^* has already been calculated by solving (2.10) numerically. In that case the method proceeds by rewriting the hedging demand in Proposition 1 as

$$X_t^* \pi_{2t}^* = -(\sigma_t')^{-1} \mathbf{E}_t[G_{t,T}] \tag{3.1}$$

where $G_{t,T} \equiv G_{t,T}^c + G_{t,T}^x$, with

$$G_{t,s}^c \equiv \int_t^s \xi_{t,v} Z_1(y^* \xi_v, v) H_{t,v}\, \mathrm{d}v \quad \text{and}$$
$$G_{t,T}^x \equiv \xi_{t,T} Z_2(y^* \xi_T, T) H_{t,T}. \tag{3.2}$$

To calculate $X_t^* \pi_{2t}^*$ write the random variables in the hedges in the form of a joint system $V_{t,s}' \equiv (Y_s', \mathrm{vec}(\mathcal{D}_t Y_s)', K_{t,s}, H_{t,s}', (G_{t,s}^c)')$, where $\mathrm{vec}(\cdot)$ denotes the operator stacking the columns of a matrix one below the other, and where

$$K_{t,v} \equiv \int_t^v \left(r_s + \frac{1}{2}\theta_s'\theta_s\right) \mathrm{d}s + \int_t^v \theta_s'\, \mathrm{d}W_s,$$

$$H'_{t,v} \equiv \int_t^v \partial r(s, Y_s)\mathcal{D}_t Y_s \, \mathrm{d}s + \int_t^v \theta'_s \partial\theta(s, Y_s)\mathcal{D}_t Y_s \, \mathrm{d}s + \int_t^v \mathrm{d}W'_s \cdot \partial\theta(s, Y_s)\mathcal{D}_t Y_s$$

and $\xi_{t,v} = \exp(-K_{t,v})$. An application of Itô's lemma shows that

$$\mathrm{d}K_{t,s} = \left(r_s + \frac{1}{2}\theta'_s\theta_s\right)\mathrm{d}s + \theta'_s \, \mathrm{d}W_s, \tag{3.3}$$

$$\mathrm{d}H'_{t,s} = \partial r(s, Y_s)\mathcal{D}_t Y_s \, \mathrm{d}s + \big(\mathrm{d}W_s + \theta(s, Y_s)\, \mathrm{d}s\big)'\partial\theta(s, Y_s)\mathcal{D}_t Y_s, \tag{3.4}$$

$$\mathrm{d}G^c_{t,s} = \xi_{t,s} Z_1(y^*\xi_s, s) H_{t,s} \, \mathrm{d}s, \tag{3.5}$$

where $(Y_s, \mathcal{D}_t Y_s)$ satisfy (2.2), (2.17). Initial conditions are $H_{t,t} = 0_d$, $G^c_{t,t} = 0_d$, where 0_d denotes the d-dimensional null vector, and $K_{t,t} = 0$.

Next, simulate M trajectories of V using (3.3)–(3.5), (2.2) and (2.17). To do this select a discretization scheme, such as the Euler scheme, the Milshtein scheme or any other higher order procedure and let N be the number of discretization points of the time interval $[0, T]$ chosen. This simulation produces M estimates $\{V^{N,i}_{t,s}\colon s \in [t, T]\}$, $i = 1, \ldots, M$, of the trajectories $\{V_{t,s}\colon s \in [t, T]\}$. Given that y^* is already known (through prior computation) the terminal values of the simulated processes can be used to construct M estimates of the random variables $G_{t,T}$. Averaging over these M values yields the estimate

$$\widehat{X^*_t \pi^*_{2t}} = -(\sigma'_t)^{-1} \frac{1}{M} \sum_{i=1}^{M} G^{N,i}_{t,T}$$

of the hedging demand.[3]

Suppose now that the multiplier y^* is unknown. In this case a two-stage simulation procedure can be employed to calculate the hedging demands. The first stage mixes iteration and simulation to calculate y^*. Fix a candidate multiplier y. Based on this choice simulate $(K_{0,s}, F^c_{0,s})$ where

$$F^c_{0,s} = \int_0^s \xi_v I(y\xi_v, v)^+ \, \mathrm{d}v$$

in order to estimate the cost of consumption (the left-hand side of (2.10)). If the value obtained exceeds resources (initial wealth x) raise the candidate y

[3] The computation of the mean–variance component $X^*_t \pi^*_{1t}$ is carried out along the same lines. The evaluation of this demand component is straightforward due to the simple structure of the term $\mathbf{E}_t[D_{t,T}]$.

and repeat the calculation. In the opposite case reduce the candidate y. Repeat until the difference falls below some preselected threshold. The second stage parallels the procedure outlined above for the case of a known y^*.

Various procedures can be employed to accelerate the iterative search in stage 1. Schemes available include the Newton–Raphson procedure, the bracketing method, the bisection method, the secant method, the false position method, Ridder's method and the method of van Wijngaarden–Dekker–Brent (see Press et al., 1992 for details).

3.2 *The Doss transformation* (Detemple et al., 2003, 2005a)

The computation of the Malliavin derivatives in the portfolio formula can also be performed using a change of variables, commonly called a "Doss transformation." This change of variables, examined in Detemple et al. (2003, 2005a), leads to a characterization of Malliavin derivatives involving the solution of an ordinary differential equation (ODE). To simplify matters we assume $k = d$ (for more general cases see Detemple et al., 2005a).

Consider now a multivariate diffusion satisfying the restrictions

Condition 1. *The coefficients of the diffusion (2.2) have the following properties*:

1. *Differentiability*: $\mu^Y \in C([0, T] \times \mathbb{R}^d)$, $\sigma_j^Y \in C([0, T] \times \mathbb{R}^d)$,
2. *Boundedness*: $\mu^Y(t, 0)$ *and* $\sigma_j^Y(t, 0)$ *are bounded for all* $t \in [0, T]$, *and*
3. *Invertibility*:
   (a) $\partial_2 \sigma_j^Y \sigma_i^Y = \partial_2 \sigma_i^Y \sigma_j^Y$ (*i.e. the vector field generated by the columns of* σ *is abelian*),
   (b) *rank*$(\sigma) = d$, *a.e.*

Under the provisions of Condition 1 there exists an invertible function $\Gamma : [0, T] \times \mathbb{R}^d \mapsto \mathbb{R}^d$ solving the total differential equation

$$\partial_2 \Gamma(t, z) = \sigma\big(t, \Gamma(t, z)\big); \qquad \Gamma(t, 0) = 0 \quad \text{for all } t \in [0, T] \tag{3.6}$$

and a d-dimensional process Z satisfying

$$\mathrm{d}Z_v = A(v, Z_v)\,\mathrm{d}v + \mathrm{d}W_v; \qquad \Gamma(0, Z_0) = Y_0 \tag{3.7}$$

where

$$\begin{aligned} A(t, z) \equiv{}& \sigma\big(t, \Gamma(t, z)\big)^{-1} \\ &\times \left[\mu^Y(t, y) - \frac{1}{2} \sum_{j=1}^{d} \partial_y \sigma_j^Y(t, y) \sigma_j(t, y) \right]_{|y=\Gamma(t,z)} \\ &- \partial_1 \Gamma(t, z), \end{aligned} \tag{3.8}$$

such that

$$\mathcal{D}_t Y_v = \sigma\big(v, \Gamma(v, Z_v)\big) \mathcal{D}_t Z_v \quad \text{for all } v \geqslant t, \tag{3.9}$$

$$\mathrm{d}\mathcal{D}_t Z_v = \partial_2 A(v, Z_v)\mathcal{D}_t Z_v \,\mathrm{d}v; \qquad \lim_{v\to t} \mathcal{D}_t Z_v = I_d. \tag{3.10}$$

This final expression (3.9)–(3.10) for the Malliavin derivative $\mathcal{D}_t Y_v$ does not involve stochastic integrals. An Euler approximation based on (3.6)–(3.10) will therefore converge faster (see Detemple et al., 2005a, 2005c).

Property 3(a) of Condition 1 is always satisfied for univariate diffusions. For multivariate diffusions, it represents a commutativity condition. It is, in fact, the same commutativity condition that is needed to implement the Milshtein scheme in the case of multivariate diffusions, without resorting to further sub-discretizations of the time interval (see Detemple et al., 2005c for details).

The MCMD-Doss estimator for the optimal portfolio, is obtained by using (3.6)–(3.10) to calculate the components of the portfolio policy.

3.3 *Monte Carlo covariation* (Cvitanic et al., 2003)

Another simulation-based approach, proposed by Cvitanic et al. (2003), is based on an approximation of the volatility coefficient of the optimal wealth process. The optimal portfolio, being a linear transformation of the volatility of the wealth process, can be estimated from this approximation.

The limits

$$X_t^* \pi_t^{*\prime} \sigma_t = \lim_{h\to 0} \frac{1}{h} \mathbf{E}_t\big[F_{t+h,T}(W_{t+h} - W_t)'\big], \tag{3.11}$$

$$X_t^* \pi_t^{*\prime} \sigma_t = \lim_{h\to 0} \frac{1}{h} \mathbf{E}_t\left[F_{t,T} \frac{(W_{t+h} - W_t)'}{\xi_{t,t+h}}\right], \tag{3.12}$$

$$X_t^* \pi_t^{*\prime} \sigma_t = X_t^* \theta_t' + \lim_{h\to 0} \frac{1}{h} \mathbf{E}_t\big[F_{t,T}(W_{t+h} - W_t)'\big] \tag{3.13}$$

with $F_{t,T}$ as defined in (2.11), can serve as foundations for the approach (see Appendix B for derivations). Approximations of the optimal portfolio are obtained by fixing a discretization h and setting

$$X_t^* \pi_t^{*\prime} \sigma_t \simeq \frac{1}{h} \mathbf{E}_t\big[F_{t+h,T}(W_{t+h} - W_t)'\big], \tag{3.14}$$

$$X_t^* \pi_t^{*\prime} \sigma_t \simeq \frac{1}{h} \mathbf{E}_t\left[F_{t,T} \frac{(W_{t+h} - W_t)'}{\xi_{t,t+h}}\right], \tag{3.15}$$

$$X_t^* \pi_t^{*\prime} \sigma_t \simeq X_t^* \theta_t' + \frac{1}{h} \mathbf{E}_t\big[F_{t,T}(W_{t+h} - W_t)'\big]. \tag{3.16}$$

The conditional expectations on the right-hand sides of (3.14), (3.15) and (3.16) are then computed by simulation of the relevant processes and averaging over independent replications. The procedure originally developed by CGZ relies on (3.14) or (3.15). It is based on models with constant relative risk aversion and either terminal wealth [estimator (3.15)] or intermediate consumption [estimator (3.14)], but not both. These are subcases of the setting in Corollary 1 for which the multiplier can be eliminated, resulting in (2.24). Formula (3.16) is an alternative approximation that isolates the volatility of discounted wealth related to the volatility of the state price density. Implementation of these approximations for general preferences requires a preliminary stage to compute y^*.[4]

The procedure is easy to implement, as it does not require the simulation of auxiliary processes such as Malliavin derivatives. Nevertheless, it is based on an approximation (as h is fixed) of the optimal policy, and this will affect the convergence properties of the method. We refer to this method as MCC (Monte Carlo covariation). MCC estimators based on (3.14), (3.15) or (3.16) are numerically different. This difference only disappears in the limit as h vanishes.

3.4 Monte Carlo finite difference (MCFD)

The Monte Carlo finite difference (MCFD) method computes the hedging terms based on a version of the formulas (2.33)–(2.35) involving tangent processes. In essence the method calculates a tangent process by simulating the underlying process using perturbed initial values and then taking a finite difference approximation of the relevant derivative. This computation can be performed path-by-path. Conditional expectations involving tangent processes can then be calculated by averaging the random variables of interest over all the trajectories.[5]

Three versions of the formula involving tangent processes can serve as starting points for implementation. The first one consists of Equations (2.33)–(2.35) where $\Phi_{t,v}$ is expressed in terms of the tangent process $\nabla_{t,y} Y$ of the state variables. The second version consists of Equations (2.33)–(2.34) where $\Phi_{t,v} = -\nabla_{t,y}\log(\xi_{t,v})$ is expressed in terms of the variation of the log-RSPD. The last one is the general representation (2.36) based on the variation $\nabla_{t,y} F_{t,T}$ of the functional $F_{t,T}$.

Finite difference approximations of the relevant tangent processes are

$$\begin{aligned}
\nabla_{t,y_j}^{\tau_j,\alpha_j} Y_v &= \frac{1}{\tau_j}\big(Y_v(Y_t + \alpha_j \tau_j e_j) - Y_v\big(Y_t - (1-\alpha_j)\tau_j e_j\big)\big), \\
\Phi_{t,v}^{\tau_j,\alpha_j} &\equiv -\nabla_{t,y_j}^{\tau_j,\alpha_j} \log\big(\xi_{t,v}(Y)\big) = -\frac{1}{\tau_j}\big(\log\big(\xi_{t,T}(Y_t + \alpha_j \tau_j e_j)\big) \\
&\qquad - \log\big(\xi_{t,T}\big(Y_t - (1-\alpha_j)\tau_j e_j\big)\big)\big),
\end{aligned}$$

[4] Models with constant relative risk averse utility functions, but different risk aversion coefficients for the utility of terminal wealth and the utility of intermediate consumption, fall outside the scope of Corollary 1. For those settings a preliminary stage is also needed to compute the budget constraint multiplier y^*.

[5] Finite difference methods have been used extensively to solve PDEs or ODEs in applications such as option pricing and asset allocation. The interest of combining these methods with Monte Carlo simulation, to handle certain financial applications, has only been noted recently.

$$\nabla_{t,y_j}^{\tau_j,\alpha_j} F_{t,T} = \frac{1}{\tau_j}\big(F_{t,T}(Y_t + \alpha_j \tau_j e_j) - F_{t,T}\big(Y_t - (1-\alpha_j)\tau_j e_j\big)\big),$$

where $\alpha_j \in [0,1]$, $\tau_j > 0$, and $e_j \equiv [0,\ldots,0,1,0,\ldots,0]$ is the jth unit vector. Different choices of α_j result in different types of finite difference approximations. The selection $\alpha_j = 1$ corresponds to a single forward difference, $\alpha_j = 0$ to a single backward difference and $\alpha_j = 1/2$ to a central difference approximation of the tangent process of interest.

To simplify notation we write $\nabla_{t,y}^{\tau,\alpha} Y_v$, $\Phi_{t,v}^{\tau,\alpha}$, $\nabla_{t,y}^{\tau,\alpha} F_{t,T}$ for the vectors of tangent processes, where $\tau = (\tau_1,\ldots,\tau_k)$ and $\alpha = (\alpha_1,\ldots,\alpha_k)$. As $\tau \to 0$ the limits

$$\begin{aligned}
&\nabla_{t,y}^{\tau,\alpha} Y_v \to \nabla_{t,y} Y_v,\\
&\Phi_{t,v}^{\tau,\alpha} = -\nabla_{t,y}^{\tau,\alpha}\log\big(\xi_{t,v}(Y)\big) \to \Phi_{t,v} = -\nabla_{t,y}\log\big(\xi_{t,v}(Y)\big),\\
&\nabla_{t,y}^{\tau,\alpha} F_{t,T} \to \nabla_{t,y} F_{t,T}
\end{aligned}$$

hold (**P**-a.s.). Under regularity conditions permitting the exchange of limits and conditional expectations we can write

$$X_t^* \pi_{2t}^* = -\big(\sigma_t(t,Y_t)'\big)^{-1}\sigma^Y(t,Y_t)'\mathbf{E}_t\big[G_{t,T}(\Phi)\big]$$

with

$$\mathbf{E}_t\big[G_{t,T}(\Phi)\big] = \lim_{\tau\to 0}\mathbf{E}_t\big[G_{t,T}\big(\Phi_{t,v}^{\tau,\alpha}\big)\big] \tag{3.17}$$

or

$$\mathbf{E}_t\big[G_{t,T}(\Phi)\big] = \mathbf{E}_t[\nabla_{t,y} F_{t,T}] = \lim_{\tau\to 0}\mathbf{E}_t\big[\nabla_{t,y}^{\tau,\alpha} F_{t,T}\big]. \tag{3.18}$$

Writing $\Phi'_{t,v}(\nabla_{t,y} Y_v)$ for the left-hand side of (2.35) to emphasize the dependence on the tangent process $\nabla_{t,y} Y_v$ we also have

$$\Phi_{t,v} = \lim_{\tau\to 0}\Phi_{t,v}\big(\nabla_{t,y}^{\tau,\alpha} Y_v\big)$$

P-*a.s.*, leading to

$$\mathbf{E}_t\big[G_{t,T}(\Phi)\big] = \lim_{\tau\to 0}\mathbf{E}_t\big[G_{t,T}\big(\Phi_{t,v}\big(\nabla_{t,y}^{\tau,\alpha} Y_v\big)\big)\big]. \tag{3.19}$$

Finite difference approximations of the hedging term are obtained by fixing τ and approximating the conditional expectation $\mathbf{E}_t[G_{t,T}(\Phi)]$ by

$$\mathbf{E}_t\big[G_{t,T}(\Phi)\big] \simeq \mathbf{E}_t\big[G_{t,T}\big(\Phi_{t,v}\big(\nabla_{t,y}^{\tau,\alpha} Y_v\big)\big)\big], \tag{3.20}$$

$$\mathbf{E}_t\big[G_{t,T}(\Phi)\big] \simeq \mathbf{E}_t\big[G_{t,T}\big(\Phi_{t,v}^{\tau,\alpha}\big)\big], \tag{3.21}$$

$$\mathbf{E}_t\big[G_{t,T}(\Phi)\big] \simeq \mathbf{E}_t\big[\nabla_{t,y}^{\tau,\alpha} F_{t,T}\big]. \tag{3.22}$$

The difference between these approximations is that (3.20) calculates explicitly the derivative of the inverse marginal utilities, the interest rate and the market price of risk and approximates the tangent process of the state variables by a finite difference, (3.21) calculates explicitly the derivative of the inverse marginal utilities and approximates the tangent process of the logarithmic state price density by a finite difference, while (3.22) approximates the whole functional in the conditional expectation, including the marginal utilities, by a finite difference.

The numerical implementation of MCFD estimators, such as (3.20)–(3.22), is similar to the implementation of MCMD estimators. The procedure estimates conditional expectations by first simulating M replications of the random variable within the expectation and then averaging over these replications. In most parametric examples the conditional distribution of the random variable of interest is unknown. A numerical discretization scheme, such as the Euler or the Milshtein schemes, based on N discretization points can nevertheless be used to obtain a convergent approximation. The MCFD estimator is then calculated by averaging independent replications of these simulated random variables. The choice of α_j gives different types of finite difference approximations. The estimator obtained from forward differences ($\alpha_j = 1$) is the MCFFD estimator, the estimator obtained from backward differences ($\alpha_j = 0$) is the MCBFD estimator, and the estimator based on central differences ($\alpha_j = 1/2$) the MCCFD estimator. As in the case of deterministic finite difference methods (i.e. finite difference methods for ODEs or PDEs) the computational cost is greater for MCCFD than for MCFFD or MCBFD estimators. This stems from the need to simulate two auxiliary processes with forward and backward perturbed initial values for MCCFD estimators. In contrast, MCFFD and MCBFD estimators only require the simulation of one auxiliary process with either forward, or backward perturbed initial value. A subsequent section will show the effect on the convergence properties of the methods.

Like MCC estimators, MCFD estimators are based on approximations of the conditional expectation in the hedging terms. Note, in particular, that MCFD estimators can be viewed as approximate MCMD estimators where the tangent process and therefore the Malliavin derivative has been approximated by a finite difference. The quality of the approximation will therefore depend on the additional convergence parameter τ. This additional structure will also affect the asymptotic error distribution of an MCFD estimator.

The finite difference methods described above are used to compute portfolio hedging components that depend on tangent processes. Although the mean–variance portfolio component also takes the form of an expectation, it does not involve Malliavin derivatives or tangent processes. It can therefore be calculated in a standard manner by simulating the underlying processes (using some suitable discretization scheme) and computing the relevant expectation using an average over independent replications.

3.5 *Monte Carlo regression* (Brandt et al., 2005)

The last method surveyed is an approximation method developed to solve discrete time portfolio choice problems. This approach, proposed by Brandt et al. (2005), is based on the standard recursive dynamic programming algorithm. It combines Monte Carlo simulation with a Taylor series approximation of the value function and a regression-based computation of conditional expectations in order to calculate approximate "optimal" policies. The methodology applies to large-scale problems with path-dependent and nonstationary dynamics as well as arbitrary utility functions. We summarize the main steps in the context of a pure portfolio problem (without intermediate utility).

The procedure is recursive in nature. It is based on the (discrete time) Bellman equation for the value function V of the dynamic portfolio problem,

$$V_t(X_t, Z_t) = \max_{\pi_t} \mathbf{E}_t\big[V_{t+1}\big(X_t\big(\pi_t' R_{t+1}^e + R^f\big), Z_{t+1}\big)\big], \tag{3.23}$$

where X_t, is the endogenous wealth at time t, Z_t, is a vector of exogenous state variables at t, R_{t+1}^e the vector of risky assets' excess returns from t to $t+1$, R^f the return on the risk-free asset and π_t is the portfolio. To keep matters simple we follow Brandt et al. (2005) and assume a constant interest rate R^f. The first-order conditions (FOC) for the portfolio choice problem are

$$\mathbf{E}_t\big[\partial_1 V_{t+1}\big(X_t\big(\pi_t' R_{t+1}^e + R^f\big), Z_{t+1}\big)R_{t+1}^e\big] = 0, \tag{3.24}$$

where $\partial_1 V_{t+1}$ is the derivative of the value function with respect to future wealth.

There are three steps which are as follows:

Step 1: Simplify the initial problem (3.23) by expanding the value function in a Taylor series around $X_t R^f$, the value at $t+1$ of current wealth. To account for skewness and kurtosis effects Brandt et al. (2005) propose the fourth-order expansion[6]

$$\begin{aligned} V_t^a(X_t, Z_t) = \max_{\pi_t} \mathbf{E}_t\big[&V_{t+1}^a\big(X_t R^f, Z_{t+1}\big)\big] \\ &+ \mathbf{E}_t\big[\partial_1 V_{t+1}^a\big(X_t R^f, Z_{t+1}\big)\big(X_t \pi_t' R_{t+1}^e\big)\big] \\ &+ \frac{1}{2}\mathbf{E}_t\big[\partial_1^2 V_{t+1}^a\big(X_t R^f, Z_{t+1}\big)\big(X_t \pi_t' R_{t+1}^e\big)^2\big] \\ &+ \frac{1}{6}\mathbf{E}_t\big[\partial_1^3 V_{t+1}^a\big(X_t R^f, Z_{t+1}\big)\big(X_t \pi_t' R_{t+1}^e\big)^3\big] \\ &+ \frac{1}{24}\mathbf{E}_t\big[\partial_1^4 V_{t+1}^a\big(X_t R^f, Z_{t+1}\big)\big(X_t \pi_t' R_{t+1}^e\big)^4\big] \end{aligned}$$

[6] Brandt et al. (2005) report that a fourth-order expansion around $X_t R^f$ gives very accurate results for the particular problems that they considered.

where V^a is the value function for this new (approximate) problem. Let π_t^a be the solution of the approximate problem. The FOC leads to the following implicit expression for π^a,

$$\begin{aligned}\pi_t^a = & -\left\{\mathbf{E}_t\left[\partial_1^2 V_{t+1}^a\left(X_t R^f, Z_{t+1}\right) R_{t+1}^e \left(R_{t+1}^e\right)'\right] X_t^2\right\}^{-1} \\ & \times \Bigg\{\mathbf{E}_t\left[\partial_1 V_{t+1}^a\left(X_t R^f, Z_{t+1}\right) R_{t+1}^e\right] X_t \\ & + \frac{1}{2}\mathbf{E}_t\left[\partial_1^3 V_{t+1}^a\left(X_t R^f, Z_{t+1}\right)\left(\left(\pi_t^a\right)' R_{t+1}^e\right)^2 R_{t+1}^e\right] X_t^3 \\ & + \frac{1}{6}\mathbf{E}_t\left[\partial_1^4 V_{t+1}^a\left(X_t R^f, Z_{t+1}\right)\left(\left(\pi_t^a\right)' R_{t+1}^e\right)^3 R_{t+1}^e\right] X_t^4\Bigg\} \\ \equiv & -\left\{\mathbf{E}_t[B_{t+1}] X_t\right\}^{-1}\left\{\mathbf{E}_t[A_{t+1}] + \mathbf{E}_t\left[C_{t+1}\left(\pi_t^a\right)\right] X_t^2 \right. \\ & \left. + \mathbf{E}_t\left[D_{t+1}\left(\pi_t^a\right)\right] X_t^3\right\}. \end{aligned} \tag{3.25}$$

The structure of (3.25) shows that the solution depends on conditional moments involving the derivatives of the approximate value function and powers of the returns. Assume for now that these moments can be calculated by some procedure. The solution of (3.25) is then computed as follows:

(a) calculate the solution of the quadratic problem corresponding to the second-order expansion of the value function. This gives an explicit expression which can be used as an initial guess for solving (3.25),
(b) substitute this initial guess into the right-hand side of (3.25) to produce a new estimate of π^a on the left-hand side,
(c) iterate by repeating the previous step until consecutive estimates become close enough, i.e. the distance between consecutive estimates falls below some pre-selected tolerance level.

Step 2: Simulate a large number of sample paths of the vector $Y_t = [R_t^e, Z_t]$. This set of paths serves as the underlying tree for the application of a recursive procedure where the portfolio is approximated at each step, along each trajectory, by the solution of (3.25).

Step 3: Proceed recursively, along each trajectory, starting from the terminal date. To compute the approximate portfolio at date t proceed in the following manner. Suppose that approximate weights π_s^a for $s = t+1, \dots, T-1$ have been found. Terminal wealth starting from $X_t^a R^f$ at $t+1$ is

$$X_T^a = X_t^a R^f \prod_{s=t+1}^{T-1} \left(\pi_s^a R_{s+1}^e + R^f\right). \tag{3.26}$$

The coefficients in (3.25) can then be approximated by

$$A_{t+1} \approx \mathbf{E}_{t+1}\left[\partial u\big(X_T^a\big) \prod_{s=t+1}^{T-1} \big(\pi_s^a R_{s+1}^e + R^f\big)\right] R_{t+1}^e \tag{3.27}$$

in the case of A_{t+1}, and similar expressions for B_{t+1}, C_{t+1} and D_{t+1}. Let $a_{t+1} \equiv \partial u(X_T^a) \prod_{s=t+1}^{T-1} (\pi_s^a R_{s+1}^e + R^f) R_{t+1}^e$ be the random variable inside the expectation in (3.27) and define b_{t+1}, c_{t+1} and d_{t+1} in a similar manner. Then

$$\begin{aligned}\pi_t^a \approx -\big\{\mathbf{E}_t[b_{t+1}] X_t^a\big\}^{-1} \big\{&\mathbf{E}_t[a_{t+1}] + \mathbf{E}_t\big[c_{t+1}\big(\pi_t^a\big)\big]\big(X_t^a\big)^2 \\ &+ \mathbf{E}_t\big[d_{t+1}\big(\pi_t^a\big)\big]\big(X_t^a\big)^3\big\}.\end{aligned} \tag{3.28}$$

This approximation is treated as an exact equality to find π_t^a (in fact this construction produces an approximation of the approximate policy π_t^a). To calculate the conditional expectations of a, b, c, d the regression method of Longstaff and Schwartz (2001) is used. This simple approach uses regressions across the simulated paths to evaluate conditional expectations. Let y be a typical element of the vector $[a, b, c, d]$. The expectation of y_{t+1} is computed by regressing y_{t+1} on a vector of polynomial bases in the state variables Z_t so that,

$$\mathbf{E}_t[y_{t+1}] = \varphi(Z_t)' k_t,$$

where k_t is the vector of regression parameters, and the ith element of $\varphi(Z_t)$ corresponds to the ith term of a polynomial in Z_t of order K. The fitted values of this regression are used to construct estimates of the time t-conditional expectations of a_{t+1}, b_{t+1}, c_{t+1} and d_{t+1}, along each path m. Solving (3.28) produces an approximate portfolio $\pi_t^{a,m}$.

4 Asymptotic properties of portfolio estimators

This section describes the asymptotic error distributions of MCMD, MCC and MCFD portfolio estimators and discusses convergence issues for MCR. The results provided extend Detemple et al. (2005b, 2005c, 2005d) to settings with both running utility and utility of terminal wealth and to smooth utility functions outside the power (constant relative risk averse) class.

4.1 Notation and assumptions

Throughout the section utility functions are assumed to be smooth in the sense that $u, U \in \mathcal{C}^5$, the space of five-times continuously differentiable functions. In addition, marginal utilities satisfy the Inada conditions

$$\lim_{x \to 0} u'(x, t) = +\infty \quad \text{and} \quad \lim_{x \to 0} U'(x, T) = +\infty. \tag{4.1}$$

Let $\{t_n\colon n = 0, \dots, N-1\}$ be an equidistant discretization of the time interval $[t, T]$, with $\Delta \equiv t_{n+1} - t_n = (T-t)/N$. To state some of the results it proves useful to introduce the notation $\eta_v^N \equiv [Nv]/N$ for $v \in [0, T]$ if $Nv \notin \mathbb{N}$ and $\eta_v^N = v - 1/N$ otherwise, where $[Nv]$ is the integer part of Nv. With this definition sums can be written as integrals, e.g.

$$\sum_{n=0}^{N-1} f_{t_n}\Delta \equiv \int_t^T f_{\eta_v^N}\,\mathrm{d}v.$$

For empirical means write $\mathbf{E}^M[U] \equiv (\sum_{i=1}^M U^i)/M$, where the random variables U^i are i.i.d. replications of U.

Given that analytic formulas for the distributions associated with diffusions are usually unknown, a numerical scheme is required in order to approximate the solutions of SDEs. Let X^i be a random variable associated with the solution of an SDE and $X^{i,N}$ an approximation based on N discretization points. The notion of weak convergence is employed to assess the behavior of the approximation: the sequence $X^{i,N}$ is said to converge *weakly* to X^i as the number of discretization points N goes to infinity if and only if $\mathbf{E}[f(X^{i,N})] \to \mathbf{E}[f(X^i)]$ for all continuous, bounded functions $f \in \mathcal{C}_b$.

For a parsimonious representation of portfolios it also proves useful to define the shadow price of optimal wealth. This is the function $y_t^* \equiv y^*(t, X_t^*, Y_t)$ that is the unique solution of the nonlinear equation

$$X_t^* = \mathbf{E}_t\left[\int_t^T \xi_{t,v} I(y_t^*\xi_{t,v}, v)\,\mathrm{d}v + \xi_{t,T} J(y_t^*\xi_{t,T}, T)\right]. \tag{4.2}$$

The right-hand side of this equation is the present value of optimal consumption, post date t. Decreasing marginal utility and the Inada conditions (4.1) ensure that the shadow price y_t^* exists and is unique for all $X_t^* > 0$. Further, note that $y_t^* = y^*\xi_t$ where y^* corresponds to the initial shadow price of wealth defined previously.

4.2 *Expected approximation errors*

Let us now record a general result for expected approximation errors. Suppose the d_z-dimensional process Z satisfies the SDE

$$\mathrm{d}Z_t = a(Z_t)\,\mathrm{d}t + \sum_{j=1}^d b_j(Z_t)\,\mathrm{d}W_t^j; \quad Z_0 \text{ given}, \tag{4.3}$$

whose coefficients a, b satisfy Lipschitz and growth conditions so as to guarantee the existence and uniqueness of a solution. Let Z^N be the numerical solution of (4.3) based on the Euler scheme with N discretization points.

To describe the expected approximation error it is convenient to define the tangent process of the diffusion Z (see Section A.8), as

$$\nabla_{t,z} Z_v = \mathcal{E}^R \left(\int_t^{\cdot} \partial a(Z_s)\,\mathrm{d}s + \sum_{j=1}^{d} \int_t^{\cdot} \partial b_j(Z_s)\,\mathrm{d}W_s^j \right)_v,$$

where $\mathcal{E}^R(\cdot)$ is the right stochastic exponential (i.e. the solution of $d\mathcal{E}^R(M)_v = dM_v \mathcal{E}^R(M)_v$). Also for a function $f \in \mathcal{C}^3(\mathbb{R}^{d_z})$ define the random variables

$$\begin{aligned}
V_1(t,v) \equiv{}& -\nabla_{t,z} Z_v \int_t^v (\nabla_{t,z} Z_s)^{-1} \Bigg(\partial a(Z_s)\,\mathrm{d}Z_s \\
&+ \sum_{j=1}^{d} \left[\partial b_j a - \sum_{i=1}^{d} (\partial b_j)(\partial b_j) b_i \right](Z_s)\,\mathrm{d}W_s^j \Bigg) \\
&+ \nabla_{t,z} Z_v \int_t^v (\nabla_{t,z} Z_s)^{-1} \\
&\times \sum_{j=1}^{d} \left[\partial b_j \partial b_j a - \sum_{k,l=1}^{d} \partial_k (\partial_l a b_{l,j}) b_{k,j} \right](Z_s)\,\mathrm{d}s \\
&+ \nabla_{t,z} Z_v \int_t^v (\nabla_{t,z} Z_s)^{-1} \\
&\times \sum_{i,j=1}^{d} \left[\partial(\partial b_j \partial b_j b_i) b_i - \partial b_i \partial b_j \partial b_j b_i \right](Z_s)\,\mathrm{d}s
\end{aligned} \tag{4.4}$$

and

$$V_2(t,v) \equiv -\int_t^v \sum_{i,j=1}^{d} \nu_{i,j}(s,v)\,\mathrm{d}s, \tag{4.5}$$

where

$$\begin{aligned}
&\nu_{i,j}(s,v) \equiv \left[h^{i,j} (\nabla_{t,z} Z)^{-1} \left[(\partial b_j) b_i \right](Z), W^i \right]_s \quad \text{with} \\
&h_t^{i,j} \equiv \mathbf{E}_t \left[\mathcal{D}_{jt} \left(\partial f(Z_T) \nabla_{t,z} Z_T e_i \right) \right]
\end{aligned} \tag{4.6}$$

and e_i is the ith unit vector. A more explicit expression for $\nu_{i,j}(s,v)$ is given in Detemple et al. (2005c). Finally, for $v \in [t,T]$, define the conditional expectations

$$K_{t,v}(Z_t) \equiv \frac{1}{2} \mathbf{E}_t \left[\partial f(Z_v) V_1(t,v) + V_2(t,v) \right], \tag{4.7}$$

$$k_{t,v}(Z_t) \equiv -\mathbf{E}_t\left[\int_t^v \left(\partial f(Z_s)\left[a + \sum_{j=1}^d (\partial b_j)b_j\right](Z_s) + \sum_{j=1}^d \left[b_j' \partial^2 f b_j\right](Z_s)\right) \mathrm{d}s\right] \tag{4.8}$$

and set

$$\kappa_{t,v}(Z_t) \equiv K_{t,v}(Z_t) + k_{t,v}(Z_t). \tag{4.9}$$

With this notation we can state the following

Proposition 3. *Let $f \in \mathcal{C}^3(\mathbb{R}^{d_z})$ be such that the uniform integrability conditions*

$$\lim_{r\to\infty} \limsup_N \mathbf{E}_t\left[\mathbf{1}_{\{\|N(f(Z_T^N)-f(Z_T))\|>r\}} N \left\| f(Z_T^N) - f(Z_T)\right\|\right] = 0, \tag{4.10}$$

$$\lim_{r\to\infty} \limsup_N \mathbf{E}_t\left[\mathbf{1}_{\{\|N\int_t^T (f(Z_{\eta_v^N}^N)-f(Z_v))\,\mathrm{d}v\|>r\}} N \times \left\|\int_t^T \left(f(Z_{\eta_v^N}^N) - f(Z_v)\right)\mathrm{d}v\right\|\right] = 0 \tag{4.11}$$

hold (**P**-*a.s.*). *Then, as $N \to \infty$,*

$$N\mathbf{E}_t\left[f(Z_T^N) - f(Z_T)\right] \to \frac{1}{2} K_{t,T}(Z_t), \tag{4.12}$$

$$N\mathbf{E}_t\left[\int_t^T f(Z_{\eta_v^N}^N)\,\mathrm{d}v - \int_t^T f(Z_v)\,\mathrm{d}v\right] \to \frac{1}{2}\int_t^T \kappa_{t,v}(Z_t)\,\mathrm{d}v \tag{4.13}$$

with $K_{t,\cdot}(Z_t)$, $\kappa_{t,\cdot}(Z_t)$ as defined in (4.7), (4.9).

This proposition provides formulas for the expected errors in the approximation of functions (or functionals) evaluated at solutions of stochastic differential equations. The expressions obtained can be viewed as probabilistic representations of the formulas in Talay and Tubaro (1990) that give the errors in terms of conditional expectations of functions solving PDEs. As will become clear below expected approximation errors appear in the second-order bias terms associated with the efficient Monte Carlo estimators of conditional expectations of functions of diffusions. The formulas in Proposition 3 can be used to estimate second-order biases and infer second-order bias corrected estimators.

4.3 Asymptotic error distribution of MCMD estimators

With y_t^* as the solution of (4.2), the MCMD portfolio estimator can be written as

$$\widehat{X_t^* \pi_t^*}^{N,M} = -(\sigma_t')^{-1}\theta_t \left(\mathbf{E}_t^M\left[g_1^{MV}\left(Z_{t,T}^N; y_t^*\right)\right] + \mathbf{E}_t^M\left[\int_t^T g_2^{MV}\left(Z_{t,\eta_v^N}^N; y_t^*\right) \mathrm{d}v\right]\right) - (\sigma_t')^{-1}\left(\mathbf{E}_t^M\left[g_1^H\left(Z_{t,T}^N; y_t^*\right)\right] + \mathbf{E}_t^M\left[\int_t^T g_2^H\left(Z_{t,\eta_v^N}^N; y_t^*\right) \mathrm{d}v\right]\right). \tag{4.14}$$

In this expression $\{Z_{t,v}^N\colon\ v \in [t,T]\}$ is a numerical approximation of the d_z-dimensional process $\{Z_{t,v}' \equiv [\xi_{t,v}, H_{t,v}', \mathrm{vec}(\mathcal{D}_t Y_v)', Y_v', v]\colon\ v \in [t,T]\}$, with $d_z = 2 + d(k+1) + k$ and H as defined in (2.16). The process $Z_{t,v}$ solves

$$\mathrm{d}Z_{t,v} = a(Z_{t,v})\,\mathrm{d}v + \sum_{j=1}^{d} b_j(Z_{t,v})\,\mathrm{d}W_v^j; \quad Z_{t,t} \text{ given.}$$

The functions $g_1^{MV}, g_1^H, g_2^{MV}, g_2^H$ are $\mathcal{C}^3$-functions that determine various portfolio components and are defined by

$$g_1^{MV}(z;y) \equiv z_1 J'(yz_1, z_5); \qquad g_1^H(z;y) \equiv z_1 J'(yz_1, z_5) z_2,$$
$$g_2^{MV}(z;y) \equiv z_1 I'(yz_1, z_5); \qquad g_2^H(z;y) \equiv z_1 I'(yz_1, z_5) z_2.$$

Close inspection reveals that g_1^{MV}, g_1^H are portfolio demand components associated with terminal wealth, while g_2^{MV}, g_2^H relate to intermediate consumption.

Each portfolio component gives rise to an error term. To study the convergence properties of the joint error define

$$e_{1,t,T}^{MV,M,N} \equiv -\left(\mathbf{E}_t^M\left[g_1^{MV}\left(Z_{t,T}^N; y_t^*\right)\right] - \mathbf{E}_t\left[g_1^{MV}(Z_{t,T}; y_t^*)\right]\right)(\sigma_t')^{-1}\theta_t, \tag{4.15}$$

$$e_{1,t,T}^{H,M,N} \equiv -(\sigma_t')^{-1}\left(\mathbf{E}_t^M\left[g_1^H\left(Z_{t,T}^N; y_t^*\right)\right] - \mathbf{E}_t\left[g_1^H(Z_{t,T}; y_t^*)\right]\right), \tag{4.16}$$

$$e_{2,t,T}^{MV,M,N} \equiv -\Bigg(\mathbf{E}_t^M\Bigg[\int_t^T g_2^{MV}\big(Z_{t,\eta_v^N}^N; y_t^*\big)\,\mathrm{d}v\Bigg] - \mathbf{E}_t\Bigg[\int_t^T g_2^{MV}(Z_{t,v}; y_t^*)\,\mathrm{d}v\Bigg]\Bigg)(\sigma_t')^{-1}\theta_t, \tag{4.17}$$

$$e_{2,t,T}^{H,M,N} \equiv -(\sigma_t')^{-1}\Bigg(\mathbf{E}_t^M\Bigg[\int_t^T g_2^{H}\big(Z_{t,\eta_v^N}^N; y_t^*\big)\,\mathrm{d}v\Bigg] - \mathbf{E}_t\Bigg[\int_t^T g_2^{H}(Z_{t,v}; y_t^*)\,\mathrm{d}v\Bigg]\Bigg). \tag{4.18}$$

For $j \in \{1, 2\}$, let $(e_{j,t,T}^{M,N})' = [(e_{j,t,T}^{MV,M,N})', (e_{j,t,T}^{H,M,N})']$ be the $1 \times 2d$ random vector of approximation errors associated with the mean–variance and hedging demands for terminal wealth ($j = 1$) and intermediate consumption ($j = 2$). Finally, let $(e_{t,T}^{M,N})' = [(e_{1,t,T}^{M,N})', (e_{2,t,T}^{M,N})']$ be the $1\times 4d$ vector that incorporates all the portfolio components. Similarly, define the $1\times 4d$ random vector $C_{t,T}' \equiv [C_{1,t,T}', C_{2,t,T}']$ where

$$C_{1,t,T}' \equiv \big[-g_1^{MV}(Z_{t,T}; y_t^*)\theta_t'\sigma_t^{-1}, -g_1^{H}(Z_{t,T}; y_t^*)'\sigma_t^{-1}\big],$$

$$C_{2,t,T}' \equiv \left[-\int_t^T g_2^{MV}(Z_{t,v}; y_t^*)\,\mathrm{d}v\,\theta_t'\sigma_t^{-1}, -\int_t^T g_2^{H}(Z_{t,v}; y_t^*)'\,\mathrm{d}v\,\sigma_t^{-1}\right]$$

are random variables involved in the various portfolio components. The random variable $C_{t,T}$ plays a critical role for the joint variance of the asymptotic error distribution.

Let $\mathbb{D}^{1,2}$ be the space of random variables for which Malliavin derivatives are defined (see Section A.3). Our next proposition describes the asymptotic behavior of the estimation error.

Proposition 4. *Suppose $g \in \mathcal{C}^3(\mathbb{R}^{d_z})$ and $g(Z_{t,v}; y_t^*) \in \mathbb{D}^{1,2}$ for all $v \in [t, T]$. Also suppose that the assumptions of Proposition 3 hold, and that*

$$\lim_{r\to\infty} \mathbf{E}_t\big[\mathbf{1}_{\{\|g_j^\alpha(Z_{t,v};y_t^*)-\mathbf{E}_t[g_j^\alpha(Z_{t,v};y_t^*)]\|>r\}} \times \big\|g_j^\alpha(Z_{t,v}; y_t^*) - \mathbf{E}_t\big[g_j^\alpha(Z_{t,v}; y_t^*)\big]\big\|^2\big] = 0 \tag{4.19}$$

for all $j \in \{1, 2\}$ and $\alpha \in \{MV, H\}$. Then, as $M \to \infty$,

$$\sqrt{M} e_{t,T}^{M,N_M} \Rightarrow \epsilon^{md} \frac{1}{2} \begin{bmatrix} -K_{1,t,T}^{MV}(Y_t; y_t^*)(\sigma_t')^{-1}\theta_t \\ -(\sigma_t')^{-1}[K_{i,1,t,T}^{H}(Y_t; y_t^*)]_{i=1,\dots,d} \\ -\int_t^T \kappa_{2,t,v}^{MV}(Y_t; y_t^*)\, \mathrm{d}v (\sigma_t')^{-1}\theta_t \\ -(\sigma_t')^{-1} \int_t^T [\kappa_{i,2,t,v}^{H}(Y_t; y_t^*)]_{i=1,\dots,d}\, \mathrm{d}v \end{bmatrix} + \begin{bmatrix} L_{1,t,T}^{MV}(Y_t; y_t^*) \\ L_{1,t,T}^{H}(Y_t; y_t^*) \\ L_{2,t,T}^{MV}(Y_t; y_t^*) \\ L_{2,t,T}^{H}(Y_t; y_t^*) \end{bmatrix}, \tag{4.20}$$

where $N_M \to \infty$, as $M \to \infty$, $\epsilon^{md} = \lim_{M\to\infty} \sqrt{M}/N_M$ and

$$L_{t,T}(Y_t; y_t^*)' \equiv \big[L_{1,t,T}^{MV}(Y_t; y_t^*)', L_{1,t,T}^{H}(Y_t; y_t^*)', L_{2,t,T}^{MV}(Y_t; y_t^*)', L_{2,t,T}^{H}(Y_t; y_t^*)'\big] \tag{4.21}$$

is the terminal value of a Gaussian martingale with (deterministic) quadratic variation and conditional variance given by

$$[L, L]_{t,T}(Y_t; y_t^*) = \int_t^T \mathbf{E}_t\big[N_s(N_s)'\big]\, \mathrm{d}s = \mathbf{VAR}_t[C_{t,T}], \tag{4.22}$$

$$N_s = \mathbf{E}_s[\mathcal{D}_s C_{t,T}]. \tag{4.23}$$

The mean–variance component associated with terminal wealth $g_1^{MV}(z; y_t^*)$ induces the second-order bias function $K_{1,t,T}^{MV}(Y_t; y_t^*)$ as defined in (4.7). The components of the d-dimensional vector of hedging terms for terminal wealth $[g_1^H(z; y_t^*)]_i$ induce the second-order bias functions $K_{i,1,t,T}^{H}(Y_t; y_t^*)$ as given in (4.7). In contrast, the mean–variance component for running consumption $g_2^{MV}(z; y_t^*)$ induces two second-order biases embedded in the function $\kappa_{2,t,v}^{MV}(Y_t; y_t^*)$ as given in (4.9). Similarly, the components of the d-dimensional vector of hedging terms for running consumption $[g_2^H(z; y_t^*)]_i$ induce the second-order bias functions $\kappa_{i,2,t,v}^{H}(Y_t; y_t^*)$ defined in (4.9).

The expression for the asymptotic error distribution (4.20) has two components. The first one depends on the expected approximation error and corresponds to the second-order bias of the estimator. To illustrate the role of the parameter ϵ^{md}, and the second-order bias, note that for $i = 1, \dots, d$ confidence intervals with coverage probability $1 - \alpha$, calculated on the basis of the Gaussian process L, are

$$\big[\psi_i^-(M, N_M, \alpha), \psi_i^+(M, N_M, \alpha)\big]$$

where

$$\psi_i^{\pm}(M, N_M, \alpha) \equiv \sqrt{M}\,\widehat{\pi_{it}^* X_t^*}^{M,N_M} \pm \Phi^{-1}(\alpha/2)\frac{\sigma_{ii}^{M,N_M}}{\sqrt{M}}$$

with Φ the cumulative Gaussian distribution function and σ_{ii}^{M,N_M} a convergent estimator of the variance of the Gaussian martingale $[L_{t,T}]_i$ in (4.21). As $M \to \infty$ the true coverage probability of this interval converges to

$$\mathbf{P}\big(\pi_{it}^* X_t^* \in \big[\psi_i^-(M, N_M, \alpha), \psi_i^+(M, N_M, \alpha)\big]\big) \to \Psi\big(\alpha, \delta_i^{md}\big), \tag{4.24}$$

with

$$\Psi(\alpha, x) \equiv \Phi\big(\Phi^{-1}\big((1-\alpha)/2\big) - x\big) - \Phi\big(\Phi^{-1}(\alpha/2) - x\big), \tag{4.25}$$

$$\delta_i^{md} \equiv \frac{1}{2}\epsilon^{md}\big[\mathbf{VAR}_t[L_{t,T}]^{-\frac{1}{2}} K_{t,T}(Y_t; y_t^*)\big]_i \tag{4.26}$$

and

$$\begin{aligned} K_{t,T}(Y_t; y_t^*) \equiv & -K_{1,t,T}^{MV}(Y_t; y_t^*)(\sigma_t')^{-1}\theta_t - (\sigma_t')^{-1} K_{1,t,T}^{H}(Y_t; y_t^*) \\ & - \int_t^T \kappa_{2,t,v}^{MV}(Y_t; y_t^*)\,\mathrm{d}v(\sigma_t')^{-1}\theta_t \\ & - (\sigma_t')^{-1}\int_t^T \kappa_{2,t,v}^{H}(Y_t; y_t^*)\,\mathrm{d}v, \end{aligned} \tag{4.27}$$

where the $d \times 1$ vectors of second-order biases associated with the hedging demands for terminal wealth $K_{1,t,v}^H$ and running consumption $\kappa_{2,t,v}^H$ are given by $K_{1,t,v}^H(Y_t; y_t^*)' \equiv [K_{1,1,t,v}^H(Y_t; y_t^*), \ldots, K_{d,1,t,v}^H(Y_t; y_t^*)]$ and $\kappa_{2,t,v}^H(Y_t; y_t^*)' \equiv [\kappa_{1,2,t,v}^H(Y_t; y_t^*), \ldots, \kappa_{d,2,t,v}^H(Y_t; y_t^*)]$.

The limit (4.24) shows that a confidence interval of nominal size α, based on L, will suffer from size distortion as it will in fact cover the true value $\pi_{it}^* X_t^*$ only with probability $\Psi(\alpha, \delta_i^{md})$ and not $1-\alpha$, as initially prescribed. The degree of size distortion is measured by the distance

$$s\big(\delta_i^{md}\big) \equiv 1 - \alpha - \Psi\big(\alpha, \delta_i^{md}\big).$$

Given that $\Psi(\alpha, \cdot)$ is strictly monotone and $\Psi(\alpha, 0) = 1-\alpha$, a confidence interval has the requested nominal size if and only if there is no second-order bias, i.e. $\delta_i^{md} = 0$. In the univariate case $d = 1$, the degree of size distortion $s(\delta_1^{md})$ is negatively related to $\mathbf{VAR}_t[L_{t,T}]$, the asymptotic variance implied by the Monte Carlo averaging procedure and positively related to $\frac{\epsilon^{md}}{2} K_{t,T}(Y_t; y_t^*)$, the second-order bias implied by the discretization scheme used for the resolution of SDEs.

When $\delta_i^{md} \neq 0$ efficiency comparisons based on the length of asymptotic confidence intervals $\psi_i^+(M, N_M, \alpha) - \psi_i^-(M, N_M, \alpha)$ are invalid, because the asymptotic coverage probability is less than the requested nominal size. Conclusions pertaining to the effects of various parameters should also be drawn with care. For instance in the univariate case, note that a reduction in the variance of an estimator has two effects. On the one hand, it reduces the length of a confidence interval. On the other hand, it also, if a second-order bias exists, increases the size distortion and therefore reduces the effective coverage probability. This trade-off also appears when the numbers of replications M and discretization points N are modified. If the variance of an estimator is reduced by increasing M, leaving N unchanged, efficiency may appear to improve when in fact the effective coverage probability decreases. Alternatively, if for a fixed budget of computation time, the number of discretization points N becomes large (thus, the number of replications M goes to zero), the resulting confidence interval of the estimator becomes free of size distortion (as $\epsilon^{md} = \lim_{M\to\infty} \sqrt{M}/N_M = 0$) but its length explodes (as $\sigma_{ii}^{M,N_M}/\sqrt{M} \to \infty$).

The trade-off between the effects of M and N implies that the efficient scheme is such that the number of Monte Carlo replications must be quadrupled whenever the number of discretization points is doubled (because $\epsilon^{md} = \lim_{M\to\infty} \sqrt{M}/N_M$). In addition, the asymptotic second-order bias has to be taken into account in order to draw valid efficiency comparisons. Methods to correct for the second-order bias require the calculation of the function K. Expressions for bias corrected estimators are provided in Detemple et al. (2005c).

A similar result applies to MCMD estimators based on the Doss transformation (see Section 3.2). The use of the Doss transformation increases the rate of convergence of the Euler scheme, but not the rate of convergence of the expected approximation error. The associated portfolio estimator converges at the same speed as the estimator based on the Euler scheme without Doss transformation, has the same asymptotic covariance matrix but a different second-order bias. Likewise, using the Milshtein scheme does not improve the rate of convergence and produces an asymptotic error distribution with the same covariance matrix. The sole modification is the expression for the second-order bias (see Detemple et al., 2005c for details).

4.4 *Asymptotic properties of MCC estimators*

MCC estimators are described in (3.14)–(3.16). In what follows we examine the error behavior for (3.16). Similar convergence results hold for estimators based on (3.14) and (3.15). With the definitions

$$F_{t,T} \equiv f_1(Z_{t,T}; y_t^*) + \int_t^T f_2(Z_{t,v}; y_t^*)\, \mathrm{d}v \tag{4.28}$$

where

$$f_1(z; y) = z_1 J(yz_1, z_5), \tag{4.29}$$

$$f_2(z;y) = z_1 I(yz_1, z_5) \tag{4.30}$$

the estimation error is $(e_{t,T}^{M,N,h})' = [(e_{1,t,T}^{M,N,h})', (e_{2,t,T}^{M,N,h})']$ with

$$e_{1,t,T}^{M,N,h} \equiv (\sigma_t')^{-1}\left(\frac{1}{h}\mathbf{E}_t^M\left[f_1\left(Z_{t,T}^N; y_t^*\right)\Delta_h W_t\right] - \left(\mathbf{E}_t\left[\mathcal{D}_t f_1(Z_{t,T};y)\right]_{|y=y_t^*}\right)'\right), \tag{4.31}$$

$$e_{2,t,T}^{M,N,h} \equiv (\sigma_t')^{-1}\left(\frac{1}{h}\mathbf{E}_t^M\left[\int_t^T f_2\left(Z_{t,\eta_v^N}^N; y_t^*\right)\mathrm{d}v\,\Delta_h W_t\right] - \left(\mathbf{E}_t\left[\mathcal{D}_t\int_t^T f_2(Z_{t,v};y)\,\mathrm{d}v\right]_{|y=y_t^*}\right)'\right), \tag{4.32}$$

and $\Delta_h W_t \equiv W_{t+h} - W_t$. The asymptotic behavior of the error is described in the next proposition.

Proposition 5. *Assume that $f_1, f_2 \in \mathcal{C}^3(\mathbb{R}^{d_z})$ and suppose that $f_i(Z_{t,v}; y_t^*) \in \mathbb{D}^{1,2}$ for $i = 1, 2$. Let $K_{1,t,v}(Y_t; y_t^*)$ be defined for f_1 as in (4.7) and $\kappa_{2,t,v}(Y_t; y_t^*)$ for f_2 as in (4.9). Define the events*

$$F_1(N,h,r) = \left\{\left\|Q_{1,t,T}^{N,h}(y_t^*) - \frac{1}{2}\mathbf{E}_t\left[Q_{1,t,T}^{N,h}(y_t^*)\right]\right\| > r\right\},$$

$$F_2(N,h,r) = \left\{\left\|\int_t^T\left(Q_{2,t,\eta_v^N}^{N,h}(y_t^*) - \frac{1}{2}\mathbf{E}_t\left[Q_{2,t,\eta_v^N}^{N,h}(y_t^*)\right]\right)\mathrm{d}v\right\| > r\right\},$$

$$G_1(h,r) = \left\{\left\|f_1(Z_{t,T};y_t^*)\frac{\Delta_h W_t}{h} - \mathbf{E}_t\left[f_1(Z_{t,T};y_t^*)\frac{\Delta_h W_t}{h}\right]\right\| > r\right\},$$

$$G_2(h,r) = \left\{\left\|\int_t^T f_2(Z_{t,v};y_t^*)\,\mathrm{d}v\frac{\Delta_h W_t}{h} - \mathbf{E}_t\left[\int_t^T f_2(Z_{t,v};y_t^*)\,\mathrm{d}v\frac{\Delta_h W_t}{h}\right]\right\| > r\right\}$$

where, for $j = 1, 2$, the processes $Q_{j,t,\cdot}^{N,h}$ are given by

$$Q_{j,t,v}^{N,h}(y_t^*) \equiv N\left(f_j\left(Z_{t,v}^N; y_t^*\right) - f_j(Z_{t,v};y_t^*)\right)\frac{\Delta_h W_t}{h}. \tag{4.33}$$

Suppose that the conditions

$$\lim_{r\to\infty}\limsup_{h,N}\mathbf{E}_t\left[\mathbf{1}_{F_1(N,h,r)}\left\|Q^{N,h}_{1,t,T}(y^*_t)-\frac{1}{2}\mathbf{E}_t\left[Q^{N,h}_{1,t,T}(y^*_t)\right]\right\|\right]=0, \tag{4.34}$$

$$\lim_{r\to\infty}\limsup_{h,N}\mathbf{E}_t\left[\mathbf{1}_{F_2(N,h,r)}\left\|\int_t^T\left(Q^{N,h}_{2,t,\eta^N_v}(y^*_t)-\frac{1}{2}\mathbf{E}_t\left[Q^{N,h}_{2,t,\eta^N_v}(y^*_t)\right]\right)\mathrm{d}v\right\|\right]=0, \tag{4.35}$$

$$\lim_{r\to\infty}\limsup_{h}\mathbf{E}_t\left[\mathbf{1}_{G_1(h,r)}\left\|f_1(Z_{t,T};y^*_t)\frac{\Delta_hW_t}{h}-\mathbf{E}_t\left[f_1(Z_{t,T};y^*_t)\frac{\Delta_hW_t}{h}\right]\right\|^2\right]=0, \tag{4.36}$$

$$\lim_{r\to\infty}\limsup_{h}\mathbf{E}_t\left[\mathbf{1}_{G_2(h,r)}\left\|\int_t^T f_2(Z_{t,v};y^*_t)\,\mathrm{d}v\frac{\Delta_hW_t}{h}-\mathbf{E}_t\left[\int_t^T f_2(Z_{t,v};y^*_t)\,\mathrm{d}v\frac{\Delta_hW_t}{h}\right]\right\|^2\right]=0 \tag{4.37}$$

hold. Then, as $M\to\infty$,

$$\begin{aligned}M^{1/3}e^{M,N_M,h_M}_{t,T}\Rightarrow{}&\varepsilon^c_1(\sigma'_t)^{-1}\left(\partial_s\mathbf{E}_s\begin{bmatrix}\mathcal{D}_sf_1(Z_{t,T};y^*_t)\\ \mathcal{D}_s\int_t^T f_2(Z_{t,v};y^*_t)\,\mathrm{d}v\end{bmatrix}\right)'_{|s=t}\\&+\varepsilon^c_2\frac{1}{2}(\sigma'_t)^{-1}\left(\mathcal{D}_t\begin{bmatrix}K_{1,t,T}(Y_t;y)\\ \int_t^T\kappa_{2,t,v}(Y_t;y)\,\mathrm{d}v\end{bmatrix}\right)'_{|y=y^*_t}\\&+\left(I_2\otimes(\sigma'_t)^{-1}\right)O_{t,T}(Y_t;y^*_t),\end{aligned} \tag{4.38}$$

where $\otimes$ *denotes the Kronecker product.*[7] *In (4.38) the convergence parameters satisfy* $N_M, 1/h_M\to\infty$ *when* $M\to\infty$, *and the constants are* $\varepsilon^c_1=\lim_{M\to\infty}M^{1/3}h_M$ *and* $\varepsilon^c_2=\lim_{M\to\infty}M^{1/3}/N_M$. *The* $2d$*-dimensional Gaussian martingale* $O_{t,T}$ *has (deterministic) quadratic variation*

$$[O,O]_{t,T}(Y_t;y^*_t)=\mathbf{E}_t\begin{bmatrix}f_1(Z_{t,T};y^*_t)^2I_d & f_1(Z_{t,T};y^*_t)\int_t^T f_2(Z_{t,v};y^*_t)\,\mathrm{d}v\,I_d\\ f_1(Z_{t,T};y^*_t)\int_t^T f_2(Z_{t,v};y^*_t)\,\mathrm{d}v\,I_d & (\int_t^T f_2(Z_{t,v};y^*_t)\,\mathrm{d}v)^2I_d\end{bmatrix}.$$

[7] The Kronecker product of an $m\times n$ matrix A and a $p\times q$ matrix B corresponds to the $mp\times nq$ matrix $A\otimes B\equiv[A_{ij}B]^{j=1,\dots,n}_{i=1,\dots,m}$.

The asymptotic error distribution has three components. The first two lines on the right-hand side of (4.38) are second-order bias terms: the first line is due to the approximation of the Brownian increment by the discrete difference $\Delta_h W_t$, the second line to the approximation of the diffusion by the solution of the discretized SDE. The last line on the right-hand side of (4.38) is a martingale component associated with the estimation of the mean by an average over independent replications.

4.5 Asymptotic properties of MCFD estimators

Recall that $Z_{4,t,v} \equiv Y_v$ and, for any functional $H_{t,v}$ of Y, let $\nabla_{t,z_{j,4}} H_{t,v}$ denote the jth element of the tangent process associated with an infinitesimal perturbation of the state variable $Y_{j,t}$. To simplify the notation define the $d \times k$ matrix process $\gamma_t' \equiv \gamma(t, Y_t)' \equiv [(\sigma')^{-1}(\sigma^Y)'](t, Z_{4,t,t})$. With these definitions, the approximation error for MCFD estimators is given by $(e_{t,T}^{M,N,\tau,\alpha})' = [(e_{1,t,T}^{M,N,\tau,\alpha})', (e_{2,t,T}^{M,N,\tau,\alpha})']$ with

$$e_{1,t,T}^{M,N,\tau,\alpha} \equiv e_{1,t,T}^{MV,M,N} + e_{1,t,T}^{H,M,N,\tau,\alpha}, \tag{4.39}$$

$$e_{2,t,T}^{M,N,\tau,\alpha} \equiv e_{2,t,T}^{MV,M,N} + e_{2,t,T}^{H,M,N,\tau,\alpha}, \tag{4.40}$$

where $e_{1,t,T}^{MV,M,N}$ is given by (4.15), $e_{2,t,T}^{MV,M,N}$ by (4.17), and the hedging term approximation errors by

$$\begin{aligned} e_{1,t,T}^{H,M,N,\tau,\alpha} \equiv \gamma_t'\big[&\mathbf{E}_t^M\big[\nabla_{t,z_{j,4}}^{\tau,\alpha} f_1\big(Z_{t,T}^N; y_t^*\big)\big] \\ &- \mathbf{E}_t\big[\nabla_{t,z_{j,4}} f_1(Z_{t,T}; y_t^*)\big]\big]_{j=1,\dots,k}, \end{aligned} \tag{4.41}$$

$$\begin{aligned} e_{2,t,T}^{H,M,N,\tau,\alpha} \equiv \gamma_t'\Bigg[&\mathbf{E}_t^M\Bigg[\nabla_{t,z_{j,4}}^{\tau,\alpha} \int_t^T f_2\big(Z_{t,\eta_v^N}^N; y_t^*\big)\,\mathrm{d}v\Bigg] \\ &- \mathbf{E}_t\Bigg[\nabla_{t,z_{j,4}} \int_t^T f_2(Z_{t,v}; y_t^*)\,\mathrm{d}v\Bigg]\Bigg]_{j=1,\dots,k}. \end{aligned} \tag{4.42}$$

For MCFD estimators, the estimators of the mean–variance components are identical to those of MCMD: their convergence properties are as described in Proposition 4. The asymptotic error behavior of the hedging component is as follows. To simplify matters, we assume that $(\tau_j, \alpha_j) = (\tau, \alpha)$ for all $j = 1, \dots, k$.

Proposition 6. *Assume that the functions $f_1, f_2 \in \mathcal{C}^3(\mathbb{R}^{d_z})$ and suppose that $f_i(Z_{t,T}; y_t^*) \in \mathbb{D}^{1,2}$ for $i = 1, 2$. Let $K_{1,t,v}(Y_t; y_t^*)$ be defined for f_1 as in (4.7) and $\kappa_{2,t,v}(Y_t; y_t^*)$ for f_2 as in (4.9). Define the events*

$$F_1^j(N, \tau, r) = \left\{\left|N\nabla_{t,z_{j,4}}^{\tau,\alpha} f_1\big(Z_{t,T}^N; y_t^*\big) - \frac{1}{2}\partial_{y_j} K_{1,t,T}(Y_t; y_t^*)\right| > r\right\},$$

$$F_2^j(N,\tau,r) = \left\{ \left| N\nabla_{t,z_{j,4}}^{\tau,\alpha} \int_t^T f_2\big(Z_{t,\eta_v^N}^N; y_t^*\big)\,\mathrm{d}v - \frac{1}{2}\int_t^T \partial_{y_j}\boldsymbol{\kappa}_{2,t,\eta_v^N}(Y_t; y_t^*)\,\mathrm{d}v \right| > r \right\},$$

$$G_1^j(\tau,r) = \big\{ \big| \nabla_{t,z_{j,4}}^{\tau,\alpha} f_1(Z_{t,T}; y_t^*) - \mathbf{E}_t\big[\nabla_{t,z_{j,4}}^{\tau,\alpha} f_1(Z_{t,T}; y_t^*)\big] \big| > r \big\},$$

$$G_2^j(\tau,r) = \left\{ \left| \nabla_{t,z_{j,4}}^{\tau,\alpha} \int_t^T f_2(Z_{t,v}; y_t^*)\,\mathrm{d}v - \mathbf{E}_t\left[\nabla_{t,z_{j,4}}^{\tau,\alpha} \int_t^T f_2(Z_{t,v}; y_t^*)\,\mathrm{d}v \right] \right| > r \right\}.$$

Suppose that the conditions

$$\lim_{r\to\infty} \limsup_{1/\tau,N} \mathbf{E}_t\left[\mathbf{1}_{F_1^j(N,\tau,r)} \left| N\nabla_{t,z_{j,4}}^{\tau,\alpha} f_1\big(Z_{t,T}^N; y_t^*\big) - \frac{1}{2}\partial_{y_j} K_{1,t,T}(Y_t; y_t^*) \right| \right] = 0, \tag{4.43}$$

$$\lim_{r\to\infty} \limsup_{1/\tau,N} \mathbf{E}_t\left[\mathbf{1}_{F_2^j(N,\tau,r)} \left| N\nabla_{t,z_{j,4}}^{\tau,\alpha} \int_t^T \left(f_2\big(Z_{t,\eta_v^N}^N; y_t^*\big) - \frac{1}{2}\partial_{y_j}\boldsymbol{\kappa}_{2,t,\eta_v^N}(Y_t; y_t^*) \right) \mathrm{d}v \right| \right] = 0, \tag{4.44}$$

$$\lim_{r\to\infty} \limsup_{1/\tau} \mathbf{E}_t\big[\mathbf{1}_{G_1^j(\tau,r)} \big| \nabla_{t,z_{j,4}}^{\tau,\alpha} f_1(Z_{t,T}; y_t^*) - \mathbf{E}_t\big[\nabla_{t,z_{j,4}}^{\tau,\alpha} f_1(Z_{t,T}; y_t^*)\big] \big|^2 \big] = 0, \tag{4.45}$$

$$\lim_{r\to\infty} \limsup_{1/\tau} \mathbf{E}_t\left[\mathbf{1}_{G_2^j(\tau,r)} \left| \nabla_{t,z_{j,4}}^{\tau,\alpha} \int_t^T f_2(Z_{t,v}; y_t^*)\,\mathrm{d}v - \mathbf{E}_t\left[\nabla_{t,z_{j,4}}^{\tau,\alpha} \int_t^T f_2(Z_{t,v}; y_t^*)\,\mathrm{d}v \right] \right|^2 \right] = 0 \tag{4.46}$$

hold, for all $j = 1, \ldots, k$. *Then, as* $M \to \infty$,

(i) *if* $\alpha = 1/2$ (*MCCFD*) *we have*

$$
\begin{aligned}
& M^{1/2} e_{t,T}^{H,M,N_M,\tau_M,\alpha} \\
& \quad \Rightarrow \frac{\varepsilon_1^{fcd}}{24} \begin{bmatrix} \gamma_t' \mathbf{E}_t [\nabla^3_{t,z_{j,4}} f_1(Z_{t,T}; y_t^*)]_{j=1,\dots,k} \\ \gamma_t' \mathbf{E}_t [\nabla^3_{t,z_{j,4}} \int_t^T f_2(Z_{t,v}; y_t^*)\, \mathrm{d}v]_{j=1,\dots,k} \end{bmatrix} \\
& \qquad + \frac{\varepsilon_2^{fcd}}{2} \begin{bmatrix} \gamma_t' [\partial_{y_j} K_{1,t,T}(Y_t; y_t^*)]_{j=1,\dots,k} \\ \gamma_t' [\partial_{y_j} \int_t^T \kappa_{2,t,\eta_v^N}(Y_t; y_t^*)\, \mathrm{d}v]_{j=1,\dots,k} \end{bmatrix} \\
& \qquad + (I_2 \otimes \gamma_t') P_{t,T}(Y_t; y_t^*),
\end{aligned}
\tag{4.47}
$$

where $N_M, 1/\tau_M \to \infty$ *when* $M \to \infty$, *and where* $\varepsilon_1^{fcd} = \lim_{M\to\infty} M^{1/4}\tau_M$ *and* $\varepsilon_2^{fcd} = \lim_{M\to\infty} M^{1/2}/N_M$,

(ii) *if* $\alpha \neq 1/2$ (*MCBFD and MCFFD*)

$$
\begin{aligned}
& M^{1/2} e_{t,T}^{H,M,N_M,\tau_M,\alpha} \\
& \quad \Rightarrow \varepsilon_1^{fd} \delta(\alpha) \begin{bmatrix} \gamma_t' \mathbf{E}_t [\nabla^2_{t,z_{j,4}} f_1(Z_{t,T}; y_t^*)]_{j=1,\dots,k} \\ \gamma_t' \mathbf{E}_t [\nabla^2_{t,z_{j,4}} \int_t^T f_2(Z_{t,v}; y_t^*)\, \mathrm{d}v]_{j=1,\dots,k} \end{bmatrix} \\
& \qquad + \frac{\varepsilon_2^{fd}}{2} \begin{bmatrix} \gamma_t' [\partial_{y_j} K_{1,t,T}(Y_t; y_t^*)]_{j=1,\dots,k} \\ \gamma_t' [\partial_{y_j} \int_t^T \kappa_{2,t,\eta_v^N}(Y_t; y_t^*)\, \mathrm{d}v]_{j=1,\dots,k} \end{bmatrix} \\
& \qquad + (I_2 \otimes \gamma_t') P_{t,T}(Y_t; y_t^*),
\end{aligned}
\tag{4.48}
$$

where $N_M, 1/\tau_M \to \infty$ *when* $M \to \infty$, *with* $\delta(\alpha) = (2\alpha - 1)/2$, *and where* $\varepsilon_1^{fd} = \lim_{M\to\infty} M^{1/2}\tau_M$ *and* $\varepsilon_2^{fd} = \lim_{M\to\infty} M^{1/2}/N_M$. *The random variable* $P_{t,T}(Y_t; y_t^*)$ *is the terminal point of a* $2d \times 1$ *dimensional Gaussian martingale with quadratic variation*

$$
[P, P]_{t,T}(Y_t; y) = \mathbf{E}_{t,Y_t}\left[\int_t^T L(v, Z_{t,v}; y) L(v, Z_{t,v}; y)'\, \mathrm{d}v\right],
$$

where

$$
L(v, Z_{t,v}; y) = \mathbf{E}_{v,Z_{t,v}}\left[\mathcal{D}_v \begin{bmatrix} (\nabla_{v,Z_{t,v}} f_1(Z_{t,T}; y))' \\ (\nabla_{v,Z_{t,v}} \int_t^T f_2(Z_{t,s}; y)\, \mathrm{d}s)' \end{bmatrix}\right].
$$

As for MCC estimators the asymptotic error distribution has three components. The first two are second-order bias terms due to the finite difference approximation of the tangent process and to the numerical approximation of SDEs characterizing the underlying diffusions. As for MCMD and MCC estimators there is a trade-off between these error components. Also, it is apparent that the convergence rate of MCFD estimators is better than that of MCC estimators. But, the nature of the differencing scheme (i.e. forward, backward or central) only affects the second-order bias. The last component, $P_{t,T}$, is the terminal point of a Gaussian martingale. It describes the asymptotic distribution of the normalized difference between the empirical mean based on random variables drawn from the true distribution of the state variables and the true conditional expectation. This component is present even if simulation from the true distribution of the state variables is feasible.

For continuous functions the speeds of convergence of MCFD and MCMD estimators are identical. But even if sampling from the true distribution is possible, MCFD estimators will suffer from an additional second-order bias term due to the finite difference approximation of the tangent process. For discontinuous functions MCFD estimators converge more slowly than MCMD estimators (see Detemple et al., 2005d): MCCFD converges at the rate $M^{-2/5}$, whereas MCFFD and MCBFD converge at the same rate as MCC, $M^{-1/3}$.

4.6 Remarks and interpretations

The asymptotic MCMD error distribution depends on the number of Monte Carlo replications M used to approximate the conditional expectation and the number of discretization points N used to approximate the random variables in the expectation. As shown by Duffie and Glynn (1995) efficient Monte Carlo estimators of conditional expectations are obtained if the parameters M, N are chosen along the diagonal $\sqrt{M}/N =$ constant of the space of convergence parameters. Efficient Monte Carlo estimators, unfortunately, have noncentered error distributions, therefore suffer from a second-order bias. As discussed in Section 4.3, second-order biases cannot be ignored when the relative efficiencies of different Monte Carlo estimators are compared. For MCMD estimators Detemple et al. (2005c) provide analytic formulas for the second-order bias and second-order bias corrected estimators. They show that second-order bias corrected estimators are asymptotically equivalent to (generally) infeasible estimators that sample from the unknown true distribution of the state variables. Propositions 5 and 6 reveal that second-order biases are even more important for MCC and MCFD, as they both depend on an additional perturbation parameter. This dependence implies additional second-order bias components that appear difficult to correct for.

It should also be noted that the analysis above treats the shadow price of wealth y_t^* as a known quantity. This is clearly not the case when preferences are nonhomothetic. In this situation a Monte Carlo method (with discretized diffusion) can be combined with a numerical fixed point scheme to estimate the shadow price. The results of Proposition 3 show that the error associated with the estimation of y_t^* is of order $1/\sqrt{M}$ as long as $\lim_{M\to\infty}\sqrt{M}/N_M = \epsilon$ for some $\epsilon \in (0, \infty)$. As MCC estimators converge at the slower rate $1/M^{1/3}$ (see Proposition 5) the asymptotic error distribution is the same for known and estimated y_t^*. In contrast, because MCMD and MCFD estimators with known y_t^* converge at the faster rate $1/\sqrt{M}$ (see Propositions 4 and 6), the approximation error due to the estimation of y_t^* will not be asymptotically negligible.

An additional second-order bias term due to the approximation error of the shadow price of wealth will appear and affect the lengths of asymptotic confidence intervals. A detailed analysis of the error distribution is beyond the scope of this review article.

4.7 Asymptotic properties of MCR estimators

The portfolio estimator of Brandt et al. (2005) induces three error terms: the remainder of the Taylor approximation of the value function, the error due to the projections of conditional expectations on a polynomial basis and the Monte Carlo error introduced by the need to simulate random variables in order to perform these projections.

The convergence behavior of the MCR portfolio estimator has yet to be studied. In contrast, convergence results for Monte Carlo methods involving projections on basis functions are available for optimal stopping problems arising in the valuation of American contingent claims (see Tsitsiklis and van Roy, 2001; Clément et al., 2002; Egloff, 2005; and Glasserman and Yu, 2004). Although related, these convergence studies do not apply directly to the setting of Brandt et al. (2005): the control in the portfolio choice problem is not a binary variable and therefore has a more complex structure than the control of an optimal stopping problem. In addition, the papers of Tsitsiklis and van Roy (2001) and Clément et al. (2002) prove convergence but do not provide a convergence rate. Like the trade-off between the number of discretization points and the number of Monte Carlo replications described in Proposition 3, there is an optimal trade-off between the number of independent replications and the number of elements in the projection basis for the polynomial estimators of Brandt et al. (2005) and Longstaff and Schwartz (2001). Glasserman and Yu (2004) provide results for optimal stopping problems involving Brownian motion and geometric Brownian motion processes. In that context they show that the number of basis functions has to grow surprisingly fast to obtain convergence. For Brownian motion the number of polynomials $K = K_M$ for which accurate estimation is possible from M replications is $O(\log M)$. For geometric Brownian motion this growth rate is $O(\sqrt{\log M})$: the number of paths has to grow (faster than) exponentially with the number of polynomials.

All these results are derived in the context of American option pricing models. There are no reasons to expect better convergence results for the more complicated asset allocation problems. Egloff (2005) shows that results can be improved when bounded basis functions are used.[8] He also shows that the approximation error scales exponentially with the number of time steps. This

[8] A similar result is obtained by Gobet et al. (2005) for regression-based Monte Carlo methods used to solve backward stochastic differential equations. They provide a full convergence analysis in terms of $\mathbf{L}^2$ errors and a central limit theorem.

suggests that the MCR error will be large even for a moderate number of rebalancing times. This conjecture appears to be supported by the simulation evidence in Detemple et al. (2005b).

5 Performance evaluation: a numerical study

5.1 Experimental setting

In order to compare the different methods we use a model with an explicit solution. In this model the investor has constant relative risk aversion R (hence Corollary 1 applies) and operates in a market with a single risky stock and the riskless asset. There is a single Brownian motion W. The interest rate r is constant and the market price of risk θ follows the Ornstein–Uhlenbeck (OU) process

$$\mathrm{d}\theta_t = A(\bar{\theta} - \theta_t)\,\mathrm{d}t + \Sigma\,\mathrm{d}W_t; \qquad \theta_0 \text{ given}, \tag{5.1}$$

where A, $\bar{\theta}$, Σ are positive constants. The stock return has constant volatility σ. The investor cares about the expected utility of terminal wealth (there is no intermediate consumption).

The closed form solution for the optimal portfolio policy can be found in Wachter (2002).[9] Assume that the determinant condition

$$\Sigma^{-2}A^2 + \rho\big(1 + 2\Sigma^{-1}A\big) \geqslant 0, \tag{5.2}$$

holds, where $\rho = 1 - 1/R$, and define the constants

$$G = -\Sigma^{-1}A - \sqrt{\Sigma^{-2}A^2 + \rho\big(1 + 2\Sigma^{-1}A\big)}$$

and $\alpha = 2(A + \Sigma G)$. The optimal stock demand is $\pi_t^* = \pi_{1t}^* + \pi_{2t}^*$ where the mean–variance demand is $\pi_{1t}^* = (1/R)(\sigma_t)^{-1}\theta_t$ and the intertemporal hedging demand is

$$\pi_{2t}^* = -\frac{\rho}{R}\big[B(t,T) + C(t,T)\theta_t\big]\Sigma\sigma^{-1},$$

with

$$B(t,T) = \frac{2(1 - \exp(-\frac{1}{2}\alpha(T-t)))^2}{\alpha(\alpha + (\rho - G)\Sigma(1 - \exp(-\alpha(T-t))))}A\bar{\theta}, \tag{5.3}$$

$$C(t,T) = \frac{1 - \exp(-\alpha(T-t))}{\alpha + (\rho - G)\Sigma(1 - \exp(-\alpha(T-t)))}. \tag{5.4}$$

[9] Wachter shows that the problem reduces to a system of Riccatti ordinary differential equations. Liu (1998) and Schroder and Skiadas (1999) show that the same reduction applies when state variables follow affine processes.

5.2 Numerical results

This section reports comparison results for MCMD, MCC, MCFD and MCR. Three versions of MCFD, with forward finite differences (MCFFD), backward finite differences (MCBFD) and central finite differences (MCCFD) are tested. Three versions of MCR are also evaluated. The first one regresses on the excess returns for the last period (MCR-lin-1), the second on the excess returns for the last two periods (MCR-lin-2), the last one on the excess returns for the last four periods (MCR-lin-3).

In order to provide conclusive evidence about the efficiency of the different Monte Carlo methods, we draw 10,000 configurations of the parameters (R, T, θ_0, r) from independent uniform distributions. For each draw and each method, relative errors and execution times are recorded. A measure of accuracy, root mean square relative error (RMSRE), and a measure of speed, inverse average time (IAT), are computed from this sample, again for each method.[10] This experiment is repeated for different discretization values N and different numbers of trajectories M. The speed-accuracy trade-off can then be graphed to evaluate the relative performances of the candidate methods.

In order to use an Euler scheme that guarantees positive state price density we discretize $\log \xi$ and calculate the SPD ξ as the exponential of the discretized logarithmic SPD. Given the difficulties encountered in implementations of higher order polynomial-regression methods (see Detemple et al., 2005b) we only focus on the linear approximations MCR-lin-1, MCR-lin-2, MCR-lin-3.

The simulation experiment is designed in the following manner. The risk aversion parameter R is drawn from a uniform distribution with domain $[0.5, 5]$, the investment horizon T from a uniform distribution over the discrete set $\{1, 2, \ldots, 5\}$, the initial MPR θ from a uniform distribution over $[0.30, 1.50]$ and the constant interest rate r from a uniform distribution over $[0.01, 0.10]$. These distributions are assumed to be independent. Each draw consists of a vector $[R, T, \theta, r]$. Errors and computation times are recorded, for each method, for the pairs $(M, N) = \{(1000, 10), (4000, 20), (9000, 30), (16000, 40)\}$. These combinations of M, N are chosen so as to quadruple M when N is doubled, leaving the ratio $\sqrt{M}/N$ constant.[11] For MCC and MCFD an auxiliary parameter has to be selected. For MCC the time step h for the initial increment of the Brownian motion is set equal to the time step $1/N$, as in Cvitanic et al. (2003). Initial MPRs for MCFD methods are perturbed by setting $\tau = 0.1$. As is the case for M and N these auxiliary parameters decrease along the efficient path, in the manner

[10] IAT is measured by the number of portfolios computed per second.

[11] The ratio $\sqrt{M}/N$ is the efficiency ratio for MCMD. Increasing M and N while maintaining this ratio constant ensures convergence to the true value without modifying the structure of the second-order bias (see Detemple et al., 2005c).

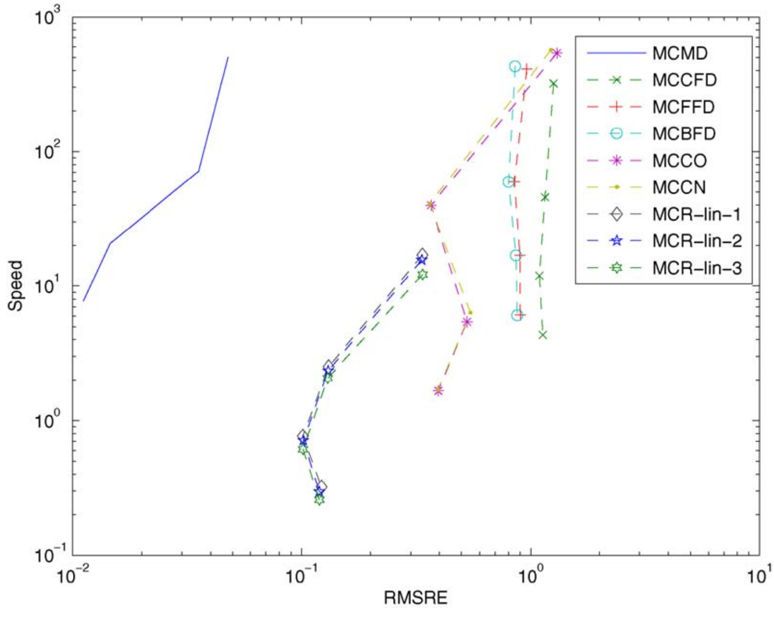

Fig. 1. This figure shows the speed-accuracy trade-off for MCMD, MCFD, MCC and MCR methods. MCCO corresponds to (3.15) and MCCN implements (3.16). All MCFD estimators are based on (3.22). Speed is measured by the inverse of the average computation time over the sample. Accuracy is measured by root mean square relative error. Four points, corresponding to the pairs $(M, N) = \{(1000, 10), (4000, 20), (9000, 30), (16000, 40)\}$, are graphed for each method. The auxiliary parameter for MCC is $h = 1/N$ and the initial auxiliary parameter for MCFD is $\tau = 0.1$. Both parameters decrease for efficient estimators as described in Propositions 5 and 6.

prescribed by the asymptotic convergence results in Propositions 5 and 6. For MCC the parameter h is cut in half if N doubles and M is multiplied by eight. For MCFD the parameter τ is cut in half if N doubles and M quadruples.

Sample statistics for RMSRE and IAT are based on 6415 "good" draws, i.e. draws for which all methods provide real results, out of the 10,000 replications. To provide perspective it is useful to note that all the "bad draws" are recorded when one of the three MCR methods fails to produce a result. Eliminating bad draws therefore advantages MCR.

Figure 1 displays the results from this experiment. The first observation is that MCMD dominates MCR, MCC and MCFD. At the same time MCR weakly dominates MCC, whereas MCC fares better than MCFD. MCMD improves on MCR by a factor in excess of 10. For a speed in the neighborhood of 10 the RMSRE of MCMD nears 10^{-2} while that of MCR is about 3×10^{-1}. Given the slopes of these trade-offs along MCMD and MCR this gap is expected to widen if M and N are further increased.

Next, we compare different versions of Monte Carlo estimators within each class.

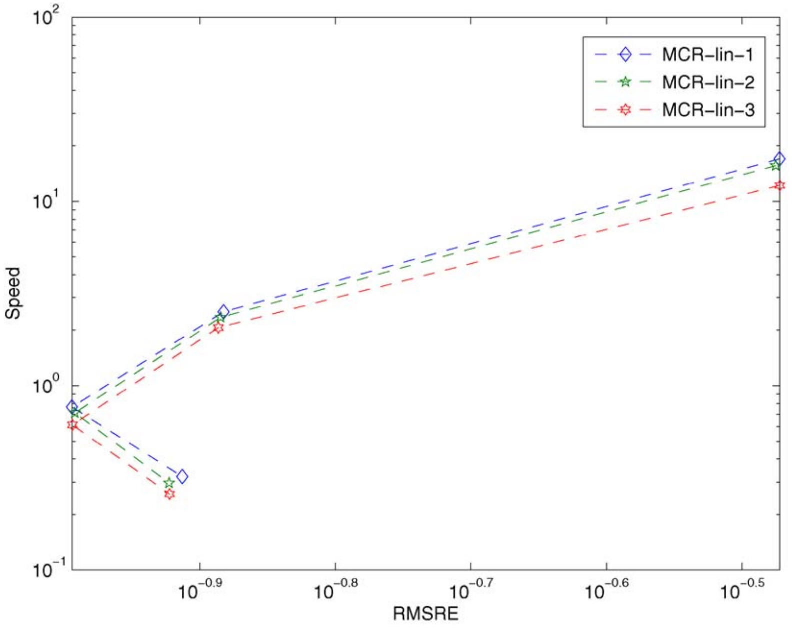

Fig. 2. This figure shows the speed-accuracy trade-off for three MCR-lin methods. Speed is measured by the inverse of the average computation time over the sample. Accuracy is measured by root mean square relative error. Four points, corresponding to the pairs $(M, N) = \{(1000, 10), (4000, 20), (9000, 30), (16000, 40)\}$, are graphed for each method.

Figure 2 illustrates that the three regression methods have a very similar performance: regressing on additional lagged returns does not improve performance. As a matter of fact it turns out that adding lagged regressors may cause the fixed point algorithm proposed by Brandt et al. (2005) to fail more frequently. Among the 10,000 configurations of $[R, T, \theta, r]$, MCR-lin-1 produced 7229 and MCR-lin-2 6984 good draws. But, in accordance with the results for American option pricing in Longstaff and Schwartz (2001), when MCR-lin provides results, the performance does not seem to depend on the choice of the orthonormal basis. This, however, is not a general property. In the present example this finding may simply be due to the fact that the true policy is linear in the MPR. MCR-lin-1 is therefore closest to the functional form of the true portfolio weight.

A similar comparison for MCC methods in Figure 3 reveals that the performance of both MCC methods is similar. Close inspection indicates that the MCC method based on (3.16) performs slightly better than the original method proposed by Cvitanic et al. (2003), based on (3.15). The RMSRE of the modified MCC method may be smaller because it estimates the hedging demand directly. In contrast, the original MCC method calculates the total portfolio weight but does not exploit the fact that for CRRA preferences the mean–variance component is known in closed form. The small size of the hedging

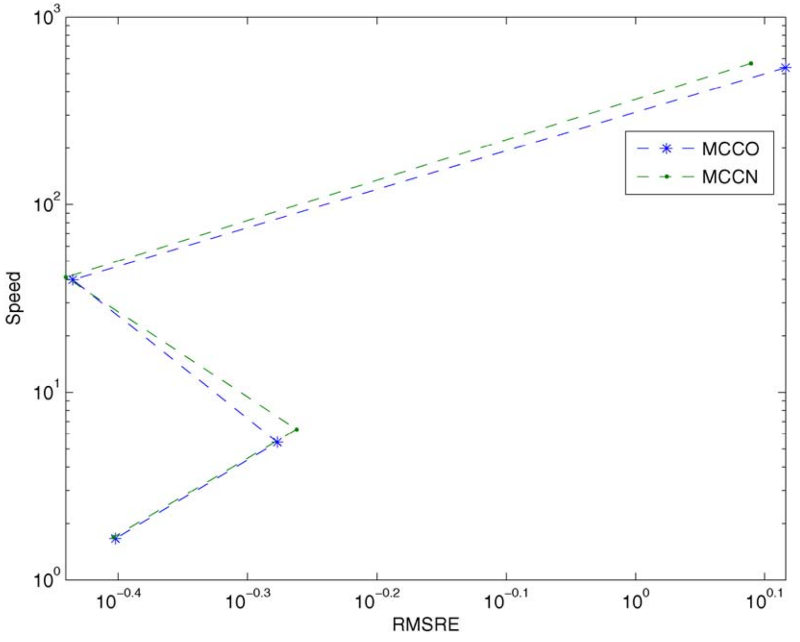

Fig. 3. This figure shows the speed-accuracy trade-off for two MCC methods. MCCO corresponds to (3.15) and MCCN implements (3.16). Speed is measured by the inverse of the average computation time over the sample. Accuracy is measured by root mean square relative error. Four points, corresponding to the triplets $(M, N, h) = \{(1000, 10, 1/10), (4000, 20, 1/20), (9000, 30, 1/30), (16000, 40, 1/40)\}$, are graphed for each method. The auxiliary time step for the initial Brownian increment h is chosen equal to the discretization step $1/N$.

demand for horizons between one and five years may be the source of the smaller relative error produced by the modified MCC method.

Finally we compare three different MCFD methods. The results in Figure 4 show that MCBFD estimators outperform both MCFFD and MCCFD estimators. MCCFD estimators are least efficient. This may be due to the fact that these estimators require the simulation of two additional perturbed processes, whereas MCFFD and MCBFD are based on a one-sided perturbation of the MPR diffusion. Hence, the computational effort to calculate MCCFD estimators is greater. At the same time Proposition 6 establishes that the speed of convergence for all methods is the same. The three MCFD estimators only differ in the second-order bias for which a ranking based on the theoretical results does not appear possible. The simulation in Figure 4 suggests that the second-order bias is larger for MCCFD than MCBFD and MCFFD.

6 Conclusion

Monte Carlo simulation is the approach of choice for high dimensional problems with large numbers of underlying variables. In contrast to alterna-

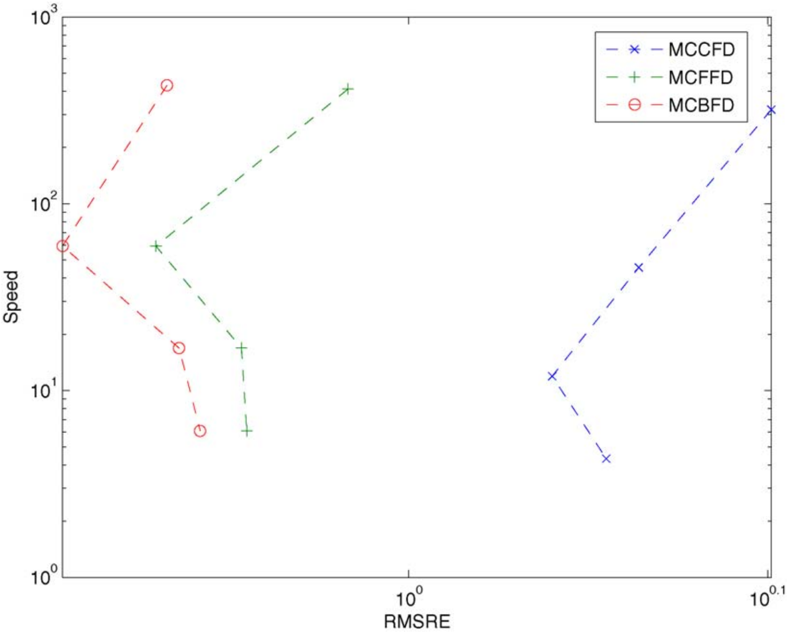

Fig. 4. This figure shows the speed-accuracy trade-off for three MCFD methods based on (3.22). Speed is measured by the inverse of the average computation time over the sample. Accuracy is measured by root mean square relative error. Four points, corresponding to the triples $(M, N, \tau) = \{(1000, 10, 1/10), (4000, 20, 1/20), (9000, 30, 1/30), (16000, 40, 1/40)\}$, are graphed for each method.

tives such as lattice methods (finite difference and finite element schemes, Markov chain approximations, quantization and quadrature schemes, etc.), simulation methods do not suffer from the well-known curse of dimensionality. As a result they emerge as natural candidates for the numerical implementation of optimal portfolio rules in high dimensional portfolio choice models. MCMD, MCC, MCFD and MCR, are various simulation schemes that have been proposed and studied during the past few years with this particular application in mind. Among these candidates, MCMD has shown a number of attractive features. One important consideration is that it is the only simulation method that attains the optimal convergence rate implied by the central limit theorem. In numerical experiments conducted it also showed superior efficiency, as measured by the trade-off between speed of computation and accuracy.

Asset allocation models with complete markets and diffusion state variables are natural candidates for the application of MCMD. In these settings the optimal portfolios can be expressed as conditional expectations of functionals of the state variables and their Malliavin derivatives, and these can be calculated numerically using Monte Carlo simulation. Settings with incomplete markets and more general forms of portfolio constraints prove more challenging. MCMD extends to these models as well, when the dual problem has an explicit solution. Constrained problems, embedding affine models, for which this can be achieved are described in Detemple and Rindisbacher (2005). Extensions of the method to more general settings, where an explicit solution to the dual is not available, remain to be carried out.

Acknowledgement

We thank MITACS for financial support.

Appendix A. An introduction to Malliavin calculus

The Malliavin calculus is a calculus of variations for stochastic processes. It applies to *Wiener (or Brownian) functionals*, i.e. random variables and stochastic processes that depend on the trajectories of Brownian motions. The Malliavin derivative, which is one element of this calculus of variations, measures the effect of a small variation in the trajectory of an underlying Brownian motion on the value of a Wiener functional.

A.1 Smooth Brownian functionals

To set the stage consider a Wiener space generated by the d-dimensional Brownian motion process $W = (W_1, \ldots, W_d)'$. As is well known we can associate each state of nature with a trajectory of the Brownian motion (the set of states of nature is the space of trajectories). Let $(t_1, \ldots, t_n)$ be a partition of the time interval $[0, T]$ and let $F(W)$ be a random variable of the form

$$F(W) \equiv f(W_{t_1}, \ldots, W_{t_n}),$$

where f is a continuously differentiable function. The random variable $F(W)$ depends (smoothly) on the d-dimensional Brownian motion W at a finite number of points along its trajectory; it is called a *smooth Brownian functional*.

A.2 The Malliavin derivative of a smooth Brownian functional

The Malliavin derivative of F is the change in F due to a change in the path of W. To simplify matters assume first that $d = 1$, i.e. there is a unique Brownian motion. Consider shifting the trajectory of W by ε starting at time t. Suppose $t_k \leqslant t < t_{k+1}$ for some $k = 1, \ldots, n-1$. The Malliavin derivative of F at t is defined by

$$\mathcal{D}_t F(W) \equiv \left. \frac{\partial f(W_{t_1} + \varepsilon \mathbf{1}_{[t,\infty[}(t_1), \ldots, W_{t_n} + \varepsilon \mathbf{1}_{[t,\infty[}(t_n))}{\partial \varepsilon} \right|_{\varepsilon=0}$$

$$= \lim_{\varepsilon \to 0} \frac{F(W + \varepsilon \mathbf{1}_{[t,\infty[}) - F(W)}{\varepsilon}, \tag{A.1}$$

where $\mathbf{1}_{[t,\infty[}$ is the indicator process of the set $[t, \infty)$ (that is $\mathbf{1}_{[t,\infty[}(s) = 1$ for $s \in [t, \infty)$; $= 0$ otherwise). In more compact form we can write

$$\mathcal{D}_t F(\omega) = \sum_{j=k}^{n} \partial_j f(W_{t_1}, \ldots, W_{t_k}, \ldots, W_{t_n}) \mathbf{1}_{[t,\infty[}(t_j), \tag{A.2}$$

where $\partial_j f$ is the derivative with respect to the jth argument of f.

A simple example will illustrate the notion. Consider the price of the stock in the Black–Scholes model. Its value at date T is given by

$$S_T = S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right) T + \sigma W_T\right),$$

where W_T is the terminal value of the univariate Brownian motion process defining the uncertainty in this model. Since $S_T = f(W_T)$ with $f(x) = S_0 \exp((\mu - \frac{1}{2}\sigma^2)T + \sigma x)$ it is clear that S_T is a smooth Brownian functional. A direct application of the definition gives

$$\mathcal{D}_t S_T = \partial f(W_T) \mathbf{1}_{[t,\infty[}(T) = \sigma S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right) T + \sigma W_T\right) = \sigma S_T.$$

In this example the stock price depends only on the Brownian motion at time T. The Malliavin derivative is then the derivative with respect to W_T. This reflects the fact that a perturbation of the path of the Brownian motion from t onward, affects S_T only through the terminal value W_T.

Suppose next that $d > 1$, i.e. the underlying Brownian motion is multidimensional. The Malliavin derivative of F at t is now a $1 \times d$-dimensional vector denoted by $\mathcal{D}_t F = (\mathcal{D}_{1t} F, \ldots, \mathcal{D}_{dt} F)$. The ith coordinate of this vector, $\mathcal{D}_{it} F$, captures the impact of a perturbation in W_i by ε starting at some time t. If $t_k \leqslant t < t_{k+1}$ we have

$$\mathcal{D}_{it} F = \sum_{j=k}^{n} \frac{\partial f}{\partial x_{ij}} (W_{t_1}, \ldots, W_{t_k}, \ldots, W_{t_n}) \mathbf{1}_{[t,\infty[}(t_j), \tag{A.3}$$

where $\partial f / \partial x_{ij}$ is the derivative with respect to the ith component of the jth argument of f (i.e. the derivative with respect to W_{it_j}).

A.3 The domain of the Malliavin derivative operator

The definition above can be extended to random variables that depend on the path of the Brownian motion over a continuous interval $[0, T]$. This extension uses the fact that a path-dependent functional can be approximated by a suitable sequence of smooth Brownian functionals. The Malliavin derivative of the path-dependent functional is then given by the limit of the Malliavin derivatives of the smooth Brownian functionals in the approximating sequence. The space of random variables for which Malliavin derivatives are defined is called $\mathbb{D}^{1,2}$. This space is the completion of the set of smooth Brownian functional with respect to the norm $\|F\|_{1,2} = (E[F^2] + \mathbf{E}[\int_0^T \|\mathcal{D}_t F\|^2\,\mathrm{d}t])^{\frac{1}{2}}$ where $\|\mathcal{D}_t F\|^2 = \sum_i (\mathcal{D}_{it} F)^2$.

A.4 Malliavin derivatives of Riemann, Wiener and Itô integrals

This extension enables us to handle stochastic integrals, which depend on the path of the Brownian motion over a continuous interval, in a very natural manner. Consider, for instance, the stochastic Wiener integral $F(W) = \int_0^T h(t)\,\mathrm{d}W_t$, where $h(t)$ is a function of time and W is one-dimensional. Integration by parts shows that $F(W) = h(T)W_T - \int_0^T W_s\,\mathrm{d}h(s)$. Straightforward calculations give

$$\begin{aligned} F(W + \varepsilon\mathbf{1}_{[t,\infty[}) - F(W) &= h(T)\big(W_T + \varepsilon\mathbf{1}_{[t,\infty[}(T)\big) \\ &\quad - \int_0^T \big(W_s + \varepsilon\mathbf{1}_{[t,\infty[}(s)\big)\,\mathrm{d}h(s) \\ &\quad - \left(h(T)W_T - \int_0^T W_s\,\mathrm{d}h(s)\right) \\ &= h(T)\varepsilon - \int_0^T \varepsilon\mathbf{1}_{[t,\infty[}(s)\,\mathrm{d}h(s) = \varepsilon h(t). \end{aligned}$$

It then follows, from the definition (A.1), that $\mathcal{D}_t F = h(t)$. The Malliavin derivative of F at t is the volatility $h(t)$ of the stochastic integral at t: this volatility measures the sensitivity of the random variable F to the Brownian innovation at t.

Next, let us consider a random Riemann integral with integrand that depends on the path of the Brownian motion. This Brownian functional takes the form $F(W) \equiv \int_0^T h_s\,\mathrm{d}s$ where h_s is a progressively measurable process (i.e. a process that depends on time and the past trajectory of the Brownian motion) such that the integral exists (i.e. $\int_0^T |h_s|\,\mathrm{d}s < \infty$ with probability one). We now have

$$F\big(W + \varepsilon\mathbf{1}[t,\infty[\big) - F(W) = \int_0^T \big(h_s\big(W + \varepsilon\mathbf{1}[t,\infty[\big) - h_s(W)\big)\,\mathrm{d}s.$$

Since $\lim_{\varepsilon\to 0}(h_s(W + \varepsilon\mathbf{1}_{[t,\infty[} - h_s(W))/\varepsilon = \mathcal{D}_t h_s(W)$ it follows that $\mathcal{D}_t F = \int_t^T \mathcal{D}_t h_s\,\mathrm{d}s$.

Finally, consider the Itô integral $F(\omega) = \int_0^T h_s(W)\,\mathrm{d}W_s$. To simplify the notation write $h^\varepsilon \equiv h(W + \varepsilon \mathbf{1}_{[t,\infty[})$ and $W^\varepsilon \equiv W + \varepsilon \mathbf{1}_{[t,\infty[}$. Integration by parts then gives

$$
\begin{aligned}
F^\varepsilon - F &= \int_0^T \left(h_s^\varepsilon - h_s\right) \mathrm{d}W_s + \int_0^T h_s^\varepsilon \,\mathrm{d}\left(W_s^\varepsilon - W_s\right) \\
&= \int_t^T \left(h_s^\varepsilon - h_s\right) \mathrm{d}W_s + h_T^\varepsilon\left(W_T^\varepsilon - W_T\right) - \int_0^T \left(W_s^\varepsilon - W_s\right) \mathrm{d}h_s^\varepsilon \\
&\quad - \int_0^T \mathrm{d}\left[W^\varepsilon - W, h^\varepsilon\right]_s \\
&= \int_t^T \left(h_s^\varepsilon - h_s\right) \mathrm{d}W_s + h_T^\varepsilon \varepsilon - \varepsilon \int_t^T \mathrm{d}h_s^\varepsilon \\
&= \int_t^T \left(h_s^\varepsilon - h_s\right) \mathrm{d}W_s + \varepsilon h_t^\varepsilon .
\end{aligned}
$$

The second equality above uses $h_s^\varepsilon = h_s$ for $s < t$ to simplify the first integral and the integration by parts formula to expand the second integral. The third equality is based on the fact that the cross-variation is null (i.e. $[W^\varepsilon - W, h^\varepsilon]_T = 0$) because $W_s^\varepsilon - W_s = \varepsilon \mathbf{1}_{[t,\infty[}(s)$ and $\mathbf{1}_{[t,\infty[}(s)$ is of bounded total variation.[12] The last equality uses, again, the integration by parts formula to simplify the last two terms. As $\lim_{\varepsilon \to 0}(h_s^\varepsilon - h_s)/\varepsilon = \mathcal{D}_t h_s$ we obtain $\mathcal{D}_t F = h_t + \int_t^T \mathcal{D}_t h_s\,\mathrm{d}W_s$.

Malliavin derivatives of Wiener, Riemann and Itô integrals depending on multidimensional Brownian motions can be defined in a similar manner. As in Section A.2 the Malliavin derivative is a d-dimensional process which can be defined component-by-component, by the operations described above.

A.5 Martingale representation and the Clark–Ocone formula

In Wiener spaces martingales with finite variances can be written as sums of Brownian increments.[13] That is, $M_t = M_0 + \int_0^t \phi_s\,\mathrm{d}W_s$ for some progressively

[12] The *total variation* of a function f is $\lim_{N\to\infty} \sum_{t_n \in \Pi^N([0,t])} |f(t_{n+1}) - f(t_n)|$ where $\Pi^N([0,t])$ is a partition with N points of the interval $[0,t]$.

[13] A *Wiener space* is the canonical probability space $(\mathcal{C}_0(\mathbb{R}_+;\mathbb{R}^d), \mathcal{B}(\mathcal{C}_0(\mathbb{R}_+;\mathbb{R}^d)), \mathbf{P})$ of nowhere differentiable functions $\mathcal{C}_0$, endowed with its Borel sigma field and the Wiener measure. The *Wiener*

measurable process ϕ, which represents the volatility coefficient of the martingale. This result is known as the martingale representation theorem. One of the most important benefits of Malliavin calculus is to identify the integrand ϕ in this representation. This is the content of the Clark–Ocone formula.

The Clark–Ocone formula states that any random variable $F \in \mathbb{D}^{1,2}$ can be decomposed as

$$F = \mathbf{E}[F] + \int_0^T \mathbf{E}_t[\mathcal{D}_t F]\,\mathrm{d}W_t, \tag{A.4}$$

where $\mathbf{E}_t[\cdot]$ is the conditional expectation at t given the information generated by the Brownian motion W. For a martingale closed by $F \in \mathbb{D}^{1,2}$ (i.e. $M_t = \mathbf{E}_t[F]$) conditional expectations can be applied to (A.4) to obtain $M_t = \mathbf{E}[F] + \int_0^t \mathbf{E}_s[\mathcal{D}_s F]\,\mathrm{d}W_s$.

An intuitive derivation of this formula can be provided along the following lines. Assume that $F \in \mathbb{D}^{1,2}$. From the martingale representation theorem we have $F = \mathbf{E}[F] + \int_0^T \phi_s\,\mathrm{d}W_s$. Taking the Malliavin derivative on each side, and applying the rules of Malliavin calculus described above, gives $\mathcal{D}_t F = \phi_t + \int_t^T \mathcal{D}_t \phi_s\,\mathrm{d}W_s$. Taking conditional expectations on each side now produces $\mathbf{E}_t[\mathcal{D}_t F] = \phi_t$ (given that $\mathbf{E}_t[\int_t^T \mathcal{D}_t \phi_s\,\mathrm{d}W_s] = 0$ and ϕ_t is known at t). Substituting this expression in the representation of F leads to (A.4).

The results above also show that the Malliavin derivative and the conditional expectation operator commute. Indeed, let $v \geqslant t$ and consider the martingale M closed by $F \in \mathbb{D}^{1,2}$. From the representations for M and F above we obtain $\mathcal{D}_t M_v = \int_t^v \mathcal{D}_t \mathbf{E}_s[\mathcal{D}_s F]\,\mathrm{d}W_s + \mathbf{E}_t[\mathcal{D}_t F]$ and $\mathcal{D}_t F = \int_t^T \mathcal{D}_t \mathbf{E}_s[\mathcal{D}_s F]\,\mathrm{d}W_s + \mathbf{E}_t[\mathcal{D}_t F]$. Taking the conditional expectation at time v of the second expression gives $\mathbf{E}_v[\mathcal{D}_t F] = \int_t^v \mathcal{D}_t \mathbf{E}_s[\mathcal{D}_s F]\,\mathrm{d}W_s + \mathbf{E}_t[\mathcal{D}_t F]$. As the formulas on the right-hand sides of these two equalities are the same we conclude that $\mathcal{D}_t M_v = \mathbf{E}_v[\mathcal{D}_t F]$. Using the definition of M_v we can also write $\mathcal{D}_t \mathbf{E}_v[F] = \mathbf{E}_v[\mathcal{D}_t F]$: the Malliavin derivative operator and the conditional expectation operator commute.

A.6 The chain rule of Malliavin calculus

In applications one often needs to compute the Malliavin derivative of a function of a path-dependent random variable. As in ordinary calculus, a chain rule also applies in the Malliavin calculus. Let $F = (F_1, \ldots, F_n)$ be a vector of random variables in $\mathbb{D}^{1,2}$ and suppose that ϕ is a differentiable function of F

measure is the measure such that the d-dimensional coordinate mapping process is a Brownian motion.

with bounded derivatives. The Malliavin derivative of $\phi(F)$ is then,

$$\mathcal{D}_t \phi(F) = \sum_{i=1}^{n} \frac{\partial \phi}{\partial x_i}(F) \mathcal{D}_t F_i$$

where $\frac{\partial \phi}{\partial x_i}(F)$ represents the derivative relative to the ith argument of ϕ.

A.7 Malliavin derivatives of stochastic differential equations

For applications to portfolio allocation it is essential to be able to calculate the Malliavin derivative of the solution of a stochastic differential equation (SDE) (i.e. the Malliavin derivative of a diffusion process). The rules of Malliavin calculus presented above can be used to that effect.

Suppose that a state variable Y_t follows the diffusion process $\mathrm{d}Y_t = \mu^Y(Y_t)\,\mathrm{d}t + \sigma^Y(Y_t)\,\mathrm{d}W_t$ where Y_0 is given and $\sigma^Y(Y_t)$ is a scalar (W is single dimensional). Equivalently, we can write the process Y in integral form as

$$Y_t = Y_0 + \int_0^t \mu^Y(Y_s)\,\mathrm{d}s + \int_0^t \sigma^Y(Y_s)\,\mathrm{d}W_s.$$

Using the results presented above, it is easy to verify that the Malliavin derivative $\mathcal{D}_t Y_s$ satisfies

$$\mathcal{D}_t Y_s = D_t Y_0 + \int_t^s \partial \mu^Y(Y_v) \mathcal{D}_t Y_v\,\mathrm{d}v + \int_t^s \partial \sigma^Y(Y_v) \mathcal{D}_t Y_v\,\mathrm{d}W_v + \sigma(Y_t).$$

As $\mathcal{D}_t Y_0 = 0$, the Malliavin derivative obeys the following linear SDE

$$\mathrm{d}(\mathcal{D}_t Y_s) = \left[\partial \mu^Y(Y_s)\,\mathrm{d}s + \partial \sigma^Y(Y_s)\,\mathrm{d}W_s\right](\mathcal{D}_t Y_s) \tag{A.5}$$

subject to the initial condition $\lim_{s \to t} \mathcal{D}_t Y_s = \sigma^Y(Y_t)$.

If $\sigma^Y(Y_t)$ is a $1 \times d$ vector (W is a d-dimensional Brownian motion) the same arguments apply to yield (A.5) subject to the initial condition $\lim_{s \to t} \mathcal{D}_t Y_s = \sigma(Y_t)$. In this multi-dimensional setting $\partial \sigma^Y(Y_s) \equiv (\partial \sigma_1^Y(Y_s), \ldots, \partial \sigma_d^Y(Y_s))$ is the row vector composed of the derivatives of the components of $\sigma^Y(Y_s)$. The Malliavin derivative $\mathcal{D}_t Y_s$ is the $1 \times d$ row vector $\mathcal{D}_t Y_s = (\mathcal{D}_{1t} Y_s, \ldots, \mathcal{D}_{dt} Y_s)$.

A.8 Stochastic flows and tangent processes

For implementation purposes it is useful to relate the Malliavin derivative to the notion of stochastic flow of a stochastic differential equation and the associated concept of a tangent process. These notions have been explored by various authors including Kunita (1986) and Malliavin (1997).

A stochastic flow of homeomorphisms (or stochastic flow for short) is an $\mathbb{R}^d$-valued random field $\{\psi_{t,v}(y,\omega)\colon\ 0 \leqslant t \leqslant v \leqslant T,\ y \in \mathbb{R}^d\}$ such that for almost all ω

(a) $\psi_{t,v}(y)$ is continuous in t, v, y,
(b) $\psi_{v,u}(\psi_{t,v}(y)) = \psi_{t,u}(y)$ for all $t \leqslant v \leqslant u$ and $y \in \mathbb{R}^d$,
(c) $\psi_{t,t}(y) = y$ for any $t \leqslant T$,
(d) the map: $\psi_{t,v} : \mathbb{R}^d \mapsto \mathbb{R}^d$ is a homeomorphism for any t, v.[14]

An important class of stochastic flows is given by the solutions of SDEs of the form

$$\mathrm{d}Y_v = \mu^Y(Y_v)\,\mathrm{d}v + \sigma^Y(Y_v)\,\mathrm{d}W_v, \quad v \in [t,T]; \quad Y_t = y.$$

The stochastic flow $\psi_{t,v}(y,\omega)$ is the position of the diffusion Y at time v, in state ω, given an initial position $Y_t = y$ at time t. A subclass of stochastic flows of homeomorphisms is obtained if $\psi_{t,v} : \mathbb{R}^d \mapsto \mathbb{R}^d$ is also required to be a diffeomorphism.[15] A element of this subclass is called a stochastic flow of diffeomorphism. For a stochastic flow of diffeomorphism determined by the solutions of an SDE, the derivative $\nabla_{t,y}\psi_{t,\cdot}(y)$ with respect to the initial condition satisfies

$$d\bigl(\nabla_{t,y}\psi_{t,v}(y)\bigr) = \left(\partial\mu^Y(Y_v)\,\mathrm{d}v + \sum_{j=1}^{d} \partial\sigma_j^Y(Y_v)\,\mathrm{d}W_v^j\right)\nabla_{t,y}\psi_{t,v}, \quad v \in [t,T], \tag{A.6}$$

subject to the initial condition $\nabla_{t,y}\psi_{t,t}(y) = I_d$. The process $\nabla_{t,y}\psi_{t,\cdot}(y)$ is called the first variation process or the tangent process.

A comparison of (A.5) and (A.6) shows that

$$\mathcal{D}_t Y_t = \mathcal{D}_t\psi_{t,v}(y) = \nabla_{t,y}\psi_{t,v}(y)\sigma^Y(y).$$

The Malliavin derivative is therefore a linear transformation of the tangent process.

Appendix B. Proofs

Proof of Proposition 1. Recall that the deflated optimal wealth process is given by $\xi_t X_t^* = \mathbf{E}_t[\int_t^T \xi_v I(y^*\xi_v, v)^+\,\mathrm{d}v + \xi_T J(y^*\xi_T, T)^+]$. Applying Itô's lemma

[14] A function is a homeomorphism if it is bijective, continuous and its inverse is also continuous.
[15] A diffeomorphism is a map between manifolds that is differentiable and has a differentiable inverse.

and the Clark–Ocone formula to this expression shows that

$$\begin{aligned}
&\xi_t X_t^* \pi_t^{*\prime} \sigma_t - \xi_t X_t^* \theta_t' \\
&\quad = -\mathbf{E}_t\left[\int_t^T \xi_v Z_1(y^* \xi_v, v)\,\mathrm{d}v + \xi_T Z_2(y^* \xi_T, T)\right]\theta_t' \\
&\qquad - \mathbf{E}_t\left[\int_t^T \xi_v Z_1(y^* \xi_v, v) H_{t,v}'\,\mathrm{d}v + \xi_T Z_2(y^* \xi_T, T) H_{t,T}'\right]
\end{aligned}$$

where

$$\begin{aligned}
Z_1(y^*\xi_v, v) &= I(y^*\xi_v, v)^+ + y^*\xi_v I'(y^*\xi_v, v) 1_{\{I(y^*\xi_v, v)\geqslant 0\}}, \\
Z_2(y^*\xi_T, T) &= J(y^*\xi_T, T)^+ + y^*\xi_T J'(y^*\xi_T, T) 1_{\{J(y^*\xi_T, T)\geqslant 0\}}, \\
H_{t,v}' &= \int_t^v (\mathcal{D}_t r_s + \theta_s' \mathcal{D}_t \theta_s)\,\mathrm{d}s + \int_t^v \mathrm{d}W_s' \cdot \mathcal{D}_t \theta_s
\end{aligned}$$

and $\mathcal{D}_t r_s$, $\mathcal{D}_t \theta_s$ are the Malliavin derivatives of the interest rate and the market price of risk. The chain rule of Malliavin calculus (Section A.6), along with the results for Malliavin derivatives of SDEs (Section A.7) now lead to (2.16) and (2.17).

From the definition of optimal wealth X^*, it also follows that

$$X_t^* - \mathbf{E}_t\left[\int_t^T \xi_{t,v} Z_1(y^*\xi_v, v)\,\mathrm{d}v + \xi_{t,T} Z_2(y^*\xi_T, T)\right] = -\mathbf{E}_t[D_{t,T}]$$

where

$$\begin{aligned}
D_{t,T} = &\int_t^T \xi_{t,v}(y^*\xi_v) I'(y^*\xi_v, v) 1_{\{I(y^*\xi_v, v)\geqslant 0\}}\,\mathrm{d}v \\
&+ \xi_{t,T}(y^*\xi_T) J'(y^*\xi_T, T) 1_{\{J(y^*\xi_T, T)\geqslant 0\}}
\end{aligned}$$

so that,

$$\begin{aligned}
X_t^* \pi_t^{*\prime} \sigma_t = &-\mathbf{E}_t[D_{t,T}]\theta_t' \\
&- \mathbf{E}_t\left[\int_t^T \xi_{t,v} Z_1(y^*\xi_v, v) H_{t,v}'\,\mathrm{d}v + \xi_{t,T} Z_2(y^*\xi_T, T) H_{t,T}'\right].
\end{aligned}$$

Transposing this formula and identifying the first term with π_1^* and the second with π_2^* leads to the formulae in the proposition. □

Proof of Corollary 1. For constant relative risk aversion $u(c,t) = \eta_t c^{1-R}/(1-R)$ and $U(X,T) = \eta_T X^{1-R}/(1-R)$, with $\eta_t \equiv \exp(-\beta t)$, we obtain the functions,

$$\begin{aligned}
&I(y\xi_v, v) = (y\xi_v/\eta_v)^{-1/R}, \qquad J(y\xi_T, T) = (y\xi_T/\eta_T)^{-1/R},\\
&y\xi_v I'(y\xi_v, v) = -(1/R)(y\xi_v/\eta_v)^{-1/R} = -(1/R)I(y\xi_v, v),\\
&y\xi_T J'(y\xi_T, T) = -(1/R)(y\xi_T/\eta_T)^{-1/R} = -(1/R)J(y\xi_T, T),\\
&Z_1(y\xi_v, v) = (1-1/R)I(y\xi_v, v),\\
&Z_2(y\xi_T, T) = (1-1/R)J(y\xi_T, T).
\end{aligned}$$

The formulas in the corollary follow by substituting these expressions in the policies of Proposition 1. □

Proof of Proposition 2. Note that the optimal consumption policy (2.18) satisfies the budget constraint

$$\mathbf{E}_t\left[\int_t^T \xi_{t,v} I(y^*\xi_t\xi_{t,v}, v)^+ \,\mathrm{d}v + \xi_{t,T} J(y^*\xi_t\xi_{t,T}, T)^+\right] = X_t^*. \tag{B.1}$$

Given the regularity conditions on preferences, the function $\mathcal{I}(t, y, Y_t)$ defined for $y > 0$ as

$$\mathcal{I}(t, y, Y_t) \equiv \mathbf{E}_t\left[\int_t^T \xi_{t,v} I(y\xi_{t,v}, v)^+ \,\mathrm{d}v + \xi_{t,T} J(y\xi_{t,T}, T)^+\right] \tag{B.2}$$

has an inverse $y^*(t, X_t, Y_t)$ that is unique and satisfies

$$\mathbf{E}_t\left[\int_t^T \xi_{t,v} I\big(y^*(t, X_t^*, Y_t)\xi_{t,v}, v\big)^+ \,\mathrm{d}v + \xi_{t,T} J\big(y^*(t, X_t^*, Y_t)\xi_{t,T}, T\big)^+\right]$$
$$= X_t^*. \tag{B.3}$$

We conclude that $y^*\xi_t = y^*(t, X_t^*, Y_t)$ **P**-a.s. Substituting the shadow price of wealth at time t, i.e. $y^*(t, X_t^*, Y_t)$, and the optimal consumption policy (2.18) in the objective function yields

$$\begin{aligned}
V(t, X_t^*, Y_t) = \mathbf{E}_t\Bigg[&\int_t^T [u \circ I^+]\big(y^*(t, X_t^*, Y_t)\xi_{t,v}, v\big)\,\mathrm{d}v\\
&+ [U \circ J^+]\big(y^*(t, X_t^*, Y_t)\xi_{t,T}, T\big)\Bigg].
\end{aligned} \tag{B.4}$$

Taking derivatives with respect to X_t^* in (B.4) and using $y^*\xi_t = y^*(t, X_t^*, Y_t)$ gives

$$V_x(t, X_t^*, Y_t) = \mathbf{E}_t[D_{t,T}]\partial_x y^*(t, X_t^*, Y_t), \tag{B.5}$$

where $D_{t,T}$ is defined in (2.21). Differentiating both sides of (B.3) with respect to wealth produces

$$\mathbf{E}_t[D_{t,T}]\frac{\partial_x y^*(t, X_t^*, Y_t)}{y^*(t, X_t^*, Y_t)} = 1, \tag{B.6}$$

so that

$$V_x(t, X_t^*, Y_t) = y^*(t, X_t^*, Y_t). \tag{B.7}$$

This establishes (2.28). Furthermore, taking logarithmic derivatives on both sides of (B.7) and using (B.6), establishes (2.29).

Finally, differentiating (B.3) with respect to the state variables and using $y^*\xi_t = y^*(t, X_t^*, Y_t)$ gives

$$\mathbf{E}_t\left[\int_t^T Z_1(y^*\xi_v, v)\nabla_{t,y}\xi_{t,v}\,\mathrm{d}v + Z_2(y^*\xi_T, T)\nabla_{t,y}\xi_{t,T}\right] + \mathbf{E}_t[D_{t,T}]\frac{\partial_y y^*(t, X_t^*, Y_t)}{y^*(t, X_t^*, Y_t)} = 0,$$

and, as $[V_{xy}/V_{xx}](t, X_t^*, Y_t) = [\partial_y y^*/\partial_x y^*](t, X_t^*, Y_t)$, with the aid of (2.29) and (B.7), we obtain

$$\frac{V_{xy}(t, X_t^*, Y_t)}{-V_{xx}(t, X_t^*, Y_t)} = \mathbf{E}_t\left[\int_t^T \xi_{t,v} Z_1(y^*\xi_v, v)\nabla_{t,y}\log\xi_{t,v}\,\mathrm{d}v + \xi_{t,T} Z_2(y^*\xi_T, T)\nabla_{t,y}\log\xi_{t,T}\right], \tag{B.8}$$

where $\nabla_{t,y}\log\xi_{t,\cdot}$ is the tangent process of $\log\xi_{t,\cdot}$ (see Appendix A). In a Markovian setting the first variation process and the Malliavin derivative are linked by $\nabla_{t,y}\log\xi_{t,v}\sigma^Y(t, Y_t) = \mathcal{D}_t\log\xi_{t,v}$ and $\mathcal{D}_t\log\xi_{t,\cdot} = -H'_{t,\cdot}$. The relation (2.30) follows. □

Proof of (3.12)–(3.13). The limits of interest are found as follows. The definition of the optimal wealth process

$$X^*_{t+h} - X^*_t + \int_t^{t+h} c^*_v \,\mathrm{d}v = \int_t^{t+h} r_v X^*_v \,\mathrm{d}v + \int_t^{t+h} X^*_v (\pi^*_v)' \big[(\mu_v - r_v 1_d)\,\mathrm{d}v + \sigma_v \,\mathrm{d}W_v\big]$$

and the Itô formula

$$\left(X^*_{t+h} - X^*_t + \int_t^{t+h} c^*_v \,\mathrm{d}v\right)(W_{t+h} - W_t)' = \int_t^{t+h} (W_v - W_t)'(\mathrm{d}X^*_v + c^*_v \,\mathrm{d}v) + \int_t^{t+h} \left(X^*_v - X^*_t + \int_t^{v} c^*_s \,\mathrm{d}s\right)\mathrm{d}W'_v + \int_t^{t+h} X^*_v (\pi^*_v)' \sigma_v \,\mathrm{d}v$$

lead to

$$\begin{aligned}
&\mathbf{E}_t\left[\left(X^*_{t+h} - X^*_t + \int_t^{t+h} c^*_v \,\mathrm{d}v\right)(W_{t+h} - W_t)'\right] \\
&= \mathbf{E}_t\left[\int_t^{t+h} (W_v - W_t)'(\mathrm{d}X^*_v + c^*_v \,\mathrm{d}v) + \int_t^{t+h} X^*_v (\pi^*_v)' \sigma_v \,\mathrm{d}v\right] \\
&= \mathbf{E}_t\left[\int_t^{t+h} \Big((W_v - W_t)'\big(r_v X^*_v + X^*_v (\pi^*_v)'(\mu_v - r_v 1_d)\big) + X^*_v (\pi^*_v)' \sigma_v\Big)\,\mathrm{d}v\right]
\end{aligned}$$

and the limit

$$\lim_{h\to 0}\frac{1}{h}\mathbf{E}_t\left[\left(X^*_{t+h}-X^*_t+\int_t^{t+h}c^*_v\,\mathrm{d}v\right)(W_{t+h}-W_t)'\right]=X^*_t(\pi^*_t)'\sigma_t. \tag{B.9}$$

Using $\mathbf{E}_t[X^*_t(W_{t+h}-W_t)']=0$,

$$\lim_{h\to 0}\frac{1}{h}\mathbf{E}_t\left[\left(\int_t^{t+h}c^*_v\,\mathrm{d}v\right)(W_{t+h}-W_t)'\right]=0$$

and

$$\begin{aligned}\mathbf{E}_t\big[X^*_{t+h}(W_{t+h}-W_t)'\big]&=\mathbf{E}_t\big[\mathbf{E}_{t+h}[F_{t+h,T}](W_{t+h}-W_t)'\big]\\&=\mathbf{E}_t\big[\mathbf{E}_{t+h}\big[F_{t+h,T}(W_{t+h}-W_t)'\big]\big]\\&=\mathbf{E}_t\big[F_{t+h,T}(W_{t+h}-W_t)'\big],\end{aligned}$$

with $F_{t+h,T}\equiv\int_{t+h}^T\xi_{t+h,v}c^*_v\,\mathrm{d}v+\xi_{t+h,T}X^*_T$, enables us to rewrite (B.9) as

$$X_t(\pi^*_t)'\sigma_t=\lim_{h\to 0}\frac{1}{h}\mathbf{E}_t\big[F_{t+h,T}(W_{t+h}-W_t)'\big]. \tag{B.10}$$

This establishes (3.14). To get (3.15) expand the coefficient $F_{t+h,T}$ as

$$\begin{aligned}F_{t+h,T}&\equiv\int_{t+h}^T\xi_{t+h,v}c^*_v\,\mathrm{d}v+\xi_{t+h,T}X^*_T\\&=\left(\int_{t+h}^T\xi_{t,v}c^*_v\,\mathrm{d}v+\xi_{t,T}X^*_T\right)\xi_{t+h,t}\\&=\left(F_{t,T}-\int_t^{t+h}\xi_{t,v}c^*_v\,\mathrm{d}v\right)\xi_{t+h,t}\end{aligned}$$

and substitute in (B.10) to write

$$\begin{aligned}X_t(\pi^*_t)'\sigma_t&=\lim_{h\to 0}\frac{1}{h}\mathbf{E}_t\big[F_{t+h,T}(W_{t+h}-W_t)'\big]\\&=\lim_{h\to 0}\frac{1}{h}\mathbf{E}_t\left[\left(F_{t,T}-\int_t^{t+h}\xi_{t,v}c^*_v\,\mathrm{d}v\right)\xi_{t+h,t}(W_{t+h}-W_t)'\right]\\&=\lim_{h\to 0}\frac{1}{h}\mathbf{E}_t\big[F_{t,T}\xi_{t+h,t}(W_{t+h}-W_t)'\big]\end{aligned}$$

$$- \lim_{h\to 0} \frac{1}{h}\mathbf{E}_t\left[\left(\int_t^{t+h} \xi_{t,v} c_v^* \, \mathrm{d}v\right)\xi_{t+h,t}(W_{t+h} - W_t)'\right]. \quad \text{(B.11)}$$

Another application of the integration by parts formula

$$\begin{aligned}
&\mathbf{E}_t\left[\left(\int_t^{t+h} \xi_{t,v} c_v^* \, \mathrm{d}v\right)\xi_{t+h,t}(W_{t+h} - W_t)'\right] \\
&\quad = \mathbf{E}_t\left[\left(\int_t^{t+h} \xi_{t,v} c_v^* \, \mathrm{d}v\right)\left(\int_t^{t+h} \xi_{v,t} \, \mathrm{d}W_v + \int_t^{t+h} (W_v - W_t)' \, \mathrm{d}\xi_{v,t} \right.\right. \\
&\qquad \left.\left. + \int_t^{t+h} \mathrm{d}[W_v', \xi_{v,t}]\right)\right]
\end{aligned}$$

shows that

$$\lim_{h\to 0} \frac{1}{h}\mathbf{E}_t\left[\left(\int_t^{t+h} \xi_{t,v} c_v^* \, \mathrm{d}v\right)\xi_{t+h,t}(W_{t+h} - W_t)'\right] = 0$$

and (B.11), therefore, becomes

$$X_t(\pi_t^*)'\sigma_t = \lim_{h\to 0} \frac{1}{h}\mathbf{E}_t\left[F_{t,T}\xi_{t+h,t}(W_{t+h} - W_t)'\right]$$

which corresponds to (3.15).

For the third expression (3.16) use the forward and backward representations of optimal wealth

$$\xi_t X_t^* + \int_0^t \xi_v c_v^* \, \mathrm{d}v = X_0 + \int_0^t \xi_v X_v^*\left((\pi_v^*)'\sigma_v - \theta_v'\right) \mathrm{d}W_v = \mathbf{E}_t[F_{0,T}]$$

to derive

$$\begin{aligned}
&\mathbf{E}_t\left[\mathbf{E}_{t+h}[F_{0,T}](W_{t+h} - W_t)'\right] \\
&\quad = \mathbf{E}_t\left[\left(X_0 + \int_0^{t+h} \xi_v X_v^*\left((\pi_v^*)'\sigma_v - \theta_v'\right) \mathrm{d}W_v\right)(W_{t+h} - W_t)'\right] \\
&\quad = \mathbf{E}_t\left[\left(\int_0^{t+h} \xi_v X_v^*\left((\pi_v^*)'\sigma_v - \theta_v'\right) \mathrm{d}W_v\right)(W_{t+h} - W_t)'\right] \\
&\quad = \mathbf{E}_t\left[\int_t^{t+h} \xi_v X_v^*\left((\pi_v^*)'\sigma_v - \theta_v'\right)' \mathrm{d}v\right].
\end{aligned}$$

From this equality and $\mathbf{E}_t[\mathbf{E}_{t+h}[F_{0,T}](W_{t+h} - W_t)] = \mathbf{E}_t[F_{0,T}(W_{t+h} - W_t)]$, it follows that

$$\lim_{h\to 0} \frac{1}{h}\mathbf{E}_t\big[F_{0,T}(W_{t+h} - W_t)\big] = \xi_t X_t^*\big(\sigma_t'(\pi_t^*) - \theta_t\big).$$

Substituting $\mathbf{E}_t[F_{0,T}(W_{t+h} - W_t)] = \xi_t \mathbf{E}_t[F_{t,T}(W_{t+h} - W_t)]$ on the left-hand side we conclude that

$$X_t^*\pi_t^* = (\sigma_t')^{-1}\left(X_t^*\theta_t + \lim_{h\to 0}\frac{1}{h}\mathbf{E}_t\big[F_{t,T}(W_{t+h} - W_t)\big]\right) \tag{B.12}$$

thereby establishing (3.16). □

Proof of Proposition 3. See Theorem 4 and Corollary 2 in Detemple et al. (2005c). □

Proof of Proposition 4. The functions g_i^α where $(i, \alpha) \in \{1, 2\} \times \{MV, H\}$ satisfy the conditions of Theorem 1 in Detemple et al. (2005d). The result follows. □

Proof of Proposition 5. The introduction of functions f_i, $i \in \{1, 2\}$, puts the problem into the setting of Theorem 2 in Detemple et al. (2005d). The proposition follows from their result. □

Proof of Proposition 6. The portfolio allocation problem is formulated so as to permit the application of Theorem 3 in Detemple et al. (2005d). The result of the proposition follows immediately. □

References

Brandt, M.W., Goyal, A., Santa-Clara, P., Stroud, J.R. (2005). A simulation approach to dynamic portfolio choice with an application to learning about return predictability. *Review of Financial Studies* 18, 831–873.

Breeden, D. (1979). An intertemporal asset pricing model with stochastic consumption and investment opportunities. *Journal of Financial Economics* 7, 265–296.

Brennan, M., Schwartz, E., Lagnado, R. (1997). Strategic asset allocation. *Journal of Economic Dynamics and Control* 21, 1377–1403.

Clément, E., Lamberton, D., Protter, P. (2002). An analysis of a least squares regression method for American option pricing. *Finance and Stochastics* 5, 449–471.

Cox, J.C., Huang, C.-f. (1989). Optimal consumption and portfolio policies when asset prices follow a diffusion process. *Journal of Economic Theory* 49, 33–83.

Cvitanic, J., Goukasian, L., Zapatero, F. (2003). Monte Carlo computation of optimal portfolio in complete markets. *Journal of Economic Dynamics and Control* 27, 971–986.

Detemple, J.B., Rindisbacher, M. (2005). Explicit solutions for a portfolio problem with incomplete markets and investment constraints. *Mathematical Finance* 15, 539–568.

Detemple, J.B., Garcia, R., Rindisbacher, M. (2003). A Monte-Carlo method for optimal portfolios. *Journal of Finance* 58, 401–446.

Detemple, J.B., Garcia, R., Rindisbacher, M. (2005a). Representation formulas for Malliavin derivatives of diffusion processes. *Finance and Stochastics* 9, 349–367.

Detemple, J.B., Garcia, R., Rindisbacher, M. (2005b). Intertemporal asset allocation: A comparison of methods. *Journal of Banking and Finance* 29, 2821–2848.

Detemple, J.B., Garcia, R., Rindisbacher, M. (2005c). Asymptotic properties of Monte Carlo estimators of diffusion processes. *Journal of Econometrics*, in press.

Detemple, J.B., Garcia, R., Rindisbacher, M. (2005d). Asymptotic properties of Monte Carlo estimators of derivatives of diffusion processes. *Management Science* 51, 1657–1675.

Duffie, D., Glynn, P. (1995). Efficient Monte Carlo simulation of security prices. *Annals of Applied Probability* 5, 897–905.

Egloff, D. (2005). Monte Carlo algorithms for optimal stopping and statistical learning. *Annals of Applied Probability* 15, 1396–1432.

Glasserman, P., Yu, B. (2004). Number of paths versus number of basis functions in American option pricing. *Annals of Applied Probability* 14, 2090–2119.

Gobet, E., Lemor, J.-P., Warin, X. (2005). A regression-based Monte-Carlo method to solve backward stochastic differential equations. *Annals of Applied Probability* 15, 2002–2172.

Karatzas, I., Shreve, S.E. (1998). *Methods of Mathematical Finance*. Springer-Verlag, New York.

Karatzas, I., Lehoczky, J.P., Shreve, S.E. (1987). Optimal portfolio and consumption decisions for a "small investor" on a finite horizon. *SIAM Journal of Control and Optimization* 25, 1557–1586.

Kunita, H. (1986). *Lectures on Stochastic Flows and Applications*. Springer-Verlag, Berlin/Heidelberg/New York/Tokyo.

Liu, J. (1998). Portfolio selection in stochastic environments. *Working paper*, Stanford University.

Longstaff, F., Schwartz, E. (2001). Valuing American options by simulation: A simple least-squares approach. *Review of Financial Studies* 14, 113–147.

Malliavin, P. (1997). *Stochastic Analysis*. Springer-Verlag, Berlin/Heidelberg/New York.

Markowitz, H. (1952). Portfolio selection. *Journal of Finance* 7, 77–91.

Merton, R.C. (1971). Optimum consumption and portfolio rules in a continuous time model. *Journal of Economic Theory* 3, 273–413.

Ocone, D., Karatzas, I. (1991). A generalized Clark representation formula, with application to optimal portfolios. *Stochastics and Stochastics Reports* 34, 187–220.

Press, W., Teukolsky, S., Vetterling, W., Flannery, B. (1992). Numerical Recipes in C: The Art of Scientific Computing, second ed. Cambridge University Press, Cambridge.

Pliska, S. (1986). A stochastic calculus model of continuous trading: Optimal portfolios. *Mathematics of Operations Research* 11, 371–382.

Schroder, M., Skiadas, C. (1999). Optimal consumption and portfolio selection with stochastic differential utility. *Journal of Economic Theory* 89, 68–126.

Talay, D., Tubaro, L. (1990). Expansion of the global error for numerical schemes solving stochastic differential equations. *Stochastic Analysis and its Application* 8, 483–509.

Tsitsiklis, J., Van Roy, B. (2001). Regression methods for pricing complex American-style options. *IEEE Transactions on Neural Networks* 12, 694–703.

Wachter, J. (2002). Portfolio and consumption decisions under mean-reverting returns: An exact solution for complete markets. *Journal of Financial and Quantitative Analysis* 37, 63–91.